UNIWERSYTET ŚLĄSKI
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Closed and connected graphs of functions; examples of connected punctiform spaces

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napisana pod kierunkiem
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Katowice 2008

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Introduction

The basic intuition of a continuous function is a line drawn from left to right without lifting one's pen off the paper. In an attempt at capturing this intuition with mathematical formalism, one may be tempted to think that being continuous is the same as having a connected graph. However, there is a discontinuous function $\psi\colon [0,1] \to \mathbb{R}$ with a connected graph: $\psi(x) = \sin(1/x)$ and $\psi(0) = 0$, which shows that the condition *connected graph* alone is not subtle enough to capture our intuition. As we look at the graph of this function we notice that it is not closed. Indeed, it turns out that the combination *closed connected graph* is a characterization of continuity for functions from the line into locally compact spaces — Corollary 19 in Chapter 3.

At this point there arises an intriguing question. Does a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with a closed connected graph have to be continuous? Although Jiři Jelinek gave a negative answer in [4], an interesting partial positive answer is the case when connectedness of the graph follows from the assumption that all x-sections are connected and one y-section is connected. This is the main result of Chapter 3 given as Theorem 21 and published in my joint paper with M. S. Wójcik [14]. (If X is locally connected, Y is connected and locally connected, Z is locally compact, $f\colon X \times Y \to Z$ has a closed graph with all x-sections continuous and one y-section continuous, then f is continuous.) Chapter 3 contains also an alternative version of the main result (*X locally connected* replaced with *Y locally compact*), Theorem 22 hitherto unpublished, and a considerably simplified version of the original core technical lemma — Lemma 15.

In search of a characterization of continuity that would appeal to our visual intuition of a *continuously drawn line* let us notice that if we are cutting across a strip of paper with a pair of scissors, we are in fact creating a continuous function: the trace of the scissors is the graph. After the whole graph is "drawn" we get two strips of paper: the set of points above the graph and the set of points below the graph, which leads us to the discovery that the combination of *the graph is connected* and *the complement of the graph is disconnected* characterizes continuity for any real-valued function defined on a connected space — Corollary 10 in Chapter 2, published in my joint paper with M. S. Wójcik [15].

Coming back to the graph of our function ψ let us observe that it is not locally connected. It turns out that a function from the real line into an arbitrary space is continuous if and only if its graph is both connected and locally connected, which is another characterization of continuity purely in terms of connectedness of the graph — Theorem 4 in Chapter 2, hitherto unpublished.

Suppose a continuous real-valued function on a connected space has a local extremum at every point without knowing whether it is a maximum or a minimum. Does it have to be constant? In general the answer is negative because there is a very nice, though nonmetrizable, connected space which admits such a nonconstant function — Example 34. However, Theorem 36 in Chapter 4 states that in the realm of metric spaces the answer is positive for the class of *separably connected spaces*, which includes all path connected spaces. (In a separably connected space every two points can be contained in a separable connected subset.)

It turns out that not all connected metric spaces are separably connected, although such spaces are very hard to find. Besides the four examples known so far, such a space can be constructed as a dense connected graph of a function from the real line into a nonseparable normed vector space. In fact, we have discovered a general method for producing dense connected graphs inside a broad class of products $X \times Y$ including any normed spaces X, Y of the cardinality of the continuum — Theorem 26 in Chapter 4.

Although a complete connected metric space failing to be separably connected has not been found, a complete *punctiform* space can be obtained as a closed connected graph of a function from the real line — Theorem 33. (A punctiform space contains no nontrivial connected compact subsets.) In Chapter 4 we also show that for a connected subset of the unit ball of a reflexive Banach space its being closed in the weak topology implies being separably connected in the weak topology — Theorem 29. The whole of Chapter 4 constitutes the draft of my joint paper with M. Morayne [8].

In Jelinek's article certain technical details were presented in a rather cursory way keeping some researchers in suspense whether this problem has been really solved, therefore in Appendix A we deliver a completely rewritten construction of Jelinek's function and a proof of its desired properties with all the controversial details handled with painstaking care to remove any doubts that Jelinek's original construction is indeed correct.

Chapter 1

Introductory Remarks

1.1 Notation and Terminology

We define *extremal functions* as real-valued functions defined on topological spaces having a local extremum at every point (without knowing whether it is a maximum or a minimum). A space is Baire if all of its nonempty open subsets are second category. We define a space to be *strongly Baire* if all of its closed subsets are Baire. A space is *totally disconnected* if singletons are its only connected subsets. A space is *punctiform* if it does not contain any nontrivial compact connected subsets. A space is *separably connected* if any two of its points can be contained in a connected separable subset. A space is *nonseparably connected* if it is connected and all of its nontrivial connected subsets are nonseparable.

If $E \subset X \times Y$ then $dom(E)$ is the projection of E onto X. We use both $|X|$ and $card(X)$ to denote the cardinality of X. When $card(X) = \kappa$, we often say that X has size κ or is of size κ. We denote the power set of X as $\mathcal{P}(X)$.

We often consider the graph of a function $F \colon X \to Y$ as a subset of the topological space $X \times Y$ or even as a topological space in its own right. For the sake of convenience, we decided to use the capital letter F rather than the usual f in order to be able to treat F as a subset of $X \times Y$ without having to write $Gr(f)$ or making any explanatory remarks on the spot. When we write that F is connected, we mean that F is a connected subset of $X \times Y$. Sometimes authors write f is connected to mean that f maps connected sets onto connected sets instead of the unambiguous f is Darboux, but we never do that.

Let X, Y be topological spaces. We say that $f \colon X \to Y$ is a *Darboux* function if and only if $f(E)$ is connected for every connected set $E \subset X$ and $f \colon X \to Y$ is a *connectivity* function if and only if the graph of $f|_E$ is connected for every connected set $E \subset X$.

1.2 Separation Axioms

Knowing that the definitions for separation axioms are not consistently used throughout the literature, for the sake of precision we decided to include the following definitions, which are equivalent with the ones given in [12].

Let X be a topological space. Then X is T_1 if and only if all singletons are closed; X is T_3 if and only if for every point $x \in X$ and every open set $U \subset X$ containing x there is an open set $V \subset X$ such that $x \in V \subset \overline{V} \subset U$; and X is T_5 if and only if for any two sets $A, B \subset X$ such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, there are two disjoint open sets $U_1, U_2 \subset X$ such that $A \subset U_1$ and $B \subset U_2$.

Notice that being T_3 or T_5 does not imply being T_1. Consider $X = \mathbb{R} \times \mathbb{N}$ with open sets of the form $U \times \mathbb{N}$, where U is an open subset of $\mathbb{R}$. Clearly X is T_3 and T_5, but $\overline{\{(x, n)\}} = \{x\} \times \mathbb{N}$, and thus no singleton is closed.

1.3 Connectedness and the T_5 Separation Axiom

A topological space is disconnected if and only if it can be written as a union of two disjoint nonempty open subsets. When we say that E is a disconnected subset of the topological space X — by treating E as a topological space with the induced topology containing those and only those sets which are of the form $E \cap G$ where G is open in X — we obtain two open subsets U, V of X giving us two relatively open subsets of E, namely $E \cap U$ and $E \cap V$, such that $E = (E \cap U) \cup (E \cap V)$, $E \cap U \neq \emptyset$ and $E \cap U \neq \emptyset$ and finally the relatively open sets are disjoint $(E \cap U) \cap (E \cap V) = \emptyset$. However, this does not imply that the sets U, V can be chosen to be disjoint too, which may be essential for certain arguments. But as long as the space X satisfies the T_5 separation axiom, the open sets U, V can be chosen to be disjoint. Fortunately, all metric spaces are T_5 — see [12].

Chapter 2

Continuity of Functions In Terms of Connectedness of The Graph

Let us take a look at the following function $\psi\colon \mathbb{R} \to \mathbb{R}$ with a connected graph and ask ourselves why it is not continuous.

$$\psi(x) = \begin{cases} \sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

As we look at the graph of this function, we notice that it is not closed. It turns out that for functions $f\colon \mathbb{R} \to \mathbb{R}^n$ continuity is equivalent to the graph being both connected and closed. However, the function $f(\cos(x), \sin(x)) = 1/x$ for $x \in (0, 2\pi]$, defined on the unit circle, is discontinuous although its graph is both closed and connected. There are also discontinuous functions $g\colon \mathbb{R}^2 \to \mathbb{R}$ and $h\colon \mathbb{R} \to l^2$ with closed connected graphs.

A second look at the graph of our function ψ reveals that it is not locally connected. One of the results of this chapter is the observation that for functions from the real line continuity can be characterized purely in terms of connectedness of the graph. Namely, a function $f\colon \mathbb{R} \to Y$ is continuous if and only if its graph is both connected and locally connected.

A careful look at the function ψ reveals that the complement of its graph is connected, while the complement of the graph of any continuous real-valued function is always disconnected. This observation leads us to the second result of this chapter, which is a characterization of continuity for real-valued functions in terms of connectedness. Namely, a function $f\colon X \to \mathbb{R}$, defined on an arbitrary connected space X, is continuous if and only if the graph is connected and its complement is disconnected.

Unfortunately, the real line cannot be reasonably replaced with a more general space in any of the two characterizations.

2.1 Connected and Locally Connected Graph

In this section it is important to bear in mind that for functions $f\colon \mathbb{R} \to Y$ having a connected graph is equivalent to being a connectivity function.

Fact 1. *Let Y be an arbitrary topological space. Let $F\colon \mathbb{R} \to Y$ be a function with a connected graph. Then F is a connectivity function.*

In general, having a connected graph is powerless and almost irrelevant when it comes to entailing connectivity. Looking at what happens at the point of discontinuity of the function $f(\cos(x), \sin(x)) = 1/x$ for $x \in (0, 2\pi]$, defined on the unit circle, serves to convince oneself that connectedness of the graph is far from entailing the Darboux property, not to mention connectivity. Naturally, every connectivity function is Darboux. However, our example (at the end of this section) of a Darboux function $f\colon (0,\infty) \to (0,\infty)$ with a totally disconnected graph shows all too clearly that the converse is not true.

Theorem 2. *Let Y be an arbitrary topological space. Let $F\colon \mathbb{R} \to Y$ be a connectivity function. Suppose that F is locally connected at $(x, F(x))$. Then F is continuous at x.*

Proof. Take any open set $V \subset Y$ containing $F(x)$. Then there is a connected set $K \subset F \cap (\mathbb{R} \times V)$ such that $(x, F(x)) \in Int_F(K)$. Suppose that $K \subset (-\infty, x] \times Y$. Then the set $\{(x, F(x))\}$ is relatively clopen in $F|_{[x,\infty)}$, which is a connected set since F is a connectivity function. This contradiction shows that $b \in dom(K)$ for some $b > x$. Similarly, $a \in dom(K)$ for some $a < x$. Since $dom(K)$ is connected, the set $U = (a, b) \subset K$ is a neighborhood of x with $F(U) \subset V$, which completes the proof that F is continuous at x. $\square$

Theorem 3. *Let X be a locally connected space. Let Y be an arbitrary topological space. Let $F\colon X \to Y$ be a connectivity function, continuous at x. Then F is locally connected at $(x, F(x))$.*

Proof. Let $E = B \times U$ be an open set containing $(x, F(x))$. Since F is continuous at x, there is an open set $G \subset B$ such that $x \in G$ and $F(G) \subset U$. Since X is locally connected, G can be assumed to be connected. Since F is a connectivity function, the set $F|_G$ is a connected relatively open subset of F contained in E, which completes the proof. $\square$

Putting together Fact 1, Theorem 2, and Theorem 3 we obtain a characterization of continuity purely in terms of connectedness of the graph.

Theorem 4. *Let Y be an arbitrary topological space. Let $F\colon \mathbb{R} \to Y$ be a function with a connected graph. Then F is continuous at x if and only if F is locally connected at $(x, F(x))$.*

It would be very hard to get rid of the real line as the domain of the function for this characterization of continuity because of the following example.

Example 5. *There is a discontinuous function $F\colon X \to [0,1]$ with a locally arcwise connected graph such that $F|_G$ is arcwise connected for every open connected set $G \subset X$, and X is a compact convex subset of the Euclidean plane, thus locally connected (and as nice as can be).*

Proof. Let $X = \{(x,y)\colon 0 \le x \le 1 \wedge 0 \le y \le x\}$. Let $F\colon X \to [0,1]$ be given by $F(x,0) = 0$ and $F(x,y) = 2xy/(x^2+y^2)$ for $y > 0$. For every $a \in [0,1]$ let $K_a = \{(x, ax)\colon x \in [0,1]\} \subset X$. Then $F(K_a) = \{0, 2a/(1+a^2)\}$. Hence F is not Darboux and F is discontinuous at $(0,0)$. However, if G is an open connected subset of X, then $F|_G$ is connected. Indeed, if $G \subset X \setminus \{0,0\}$ then $F|_G$ is connected because $F|_G$ is continuous, and if $(0,0) \in G$ then the set $F|_G = \{(0,0,0)\} \cup F|_{G\setminus\{(0,0)\}}$ is easily seen to be connected too. It remains to show that F is locally connected at $(0,0,0)$. Take any $r > 0$. Notice that $B_r = \{(x,y) \in X\colon x < r\}$ is an open convex set containing $(0,0)$. Then $F_r = F \cap (B_r \times [0,r))$ is an open subset of F containing $(0,0,0)$. We are done as soon as we show that F_r is arcwise connected. Consider the auxiliary function $g\colon [0,1] \to \mathbb{R}$ given by $g(a) = 2a/(1+a^2)$. Notice that $g(a) < r \iff a < g^{-1}(r)$. Therefore $F^{-1}([0,r)) = \bigcup\{K_a\colon 0 \le a < g^{-1}(r)\}$ is convex. Now, the set $dom(F_r) = F^{-1}([0,r)) \cap B_r$ is convex, as the intersection of two convex sets. Finally, we conclude that F_r is arcwise connected in the same way we argue that the graph of a separately continuous function is arcwise connected. $\square$

In certain arguments, functions with connected graphs defined on the real line can be replaced with Darboux functions, which is considerably weaker in view of the following example.

Example 6. *There is a Darboux function $f\colon (0,\infty) \to (0,\infty)$ whose graph is totally disconnected.*

Proof. In the first step, let us show that if $E \subset (0,\infty)$ has size $\mathfrak{c}$, then there is a surjection $f\colon E \to (0,\infty)$ such that $f(x)/x \notin \mathbb{Q}$ for every $x \in E$. Observe that if $E \subset (0,\infty)$ is uncountable, then for every $y \in (0,\infty)$ there is an $x \in E$ with $y/x \notin \mathbb{Q}$, because otherwise E would be countable. Let us well-order the set $(0,\infty) = \{y_\alpha\colon \alpha < \mathfrak{c}\}$. We will be constructing one by one points $(x_\alpha, f(x_\alpha))$ of the graph by transfinite induction over $\alpha < \mathfrak{c}$. For y_0 choose some $x_0 \in E$ with $y_0/x_0 \notin \mathbb{Q}$ and put $f(x_0) = y_0$. Now, for some ordinal $\beta < \mathfrak{c}$, the set $E \setminus \{x_\alpha\colon \alpha < \beta\}$ is of size $\mathfrak{c}$, so we can choose some $x_\beta \in E \setminus \{x_\alpha\colon \alpha < \beta\}$ with $y_\beta/x_\beta \notin \mathbb{Q}$ and put $f(x_\beta) = y_\beta$. Should the set $E \setminus \{x_\alpha\colon \alpha < \mathfrak{c}\}$ remain nonempty after this construction is completed, put $f(x) = \sqrt{2}$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise.

Let $\{(a_n, b_n)\colon n \in \mathbb{N}\}$ be a basis for the topology of $(0,\infty)$. Let C_1 be a Cantor set embedded in the open interval (a_1, b_1). Let $f\colon C_1 \to (0,\infty)$ be a surjection such that $f(x)/x \notin \mathbb{Q}$ for every $x \in C_1$. Now, for $n+1 \in \mathbb{N}$, the set $(a_{n+1}, b_{n+1}) \setminus \bigcup_{k+1}^{n} C_k$ contains an interval (c,d) because a finite union of Cantor sets is nowhere dense. Let C_{n+1} be a Cantor set embedded in (c,d) and let $f\colon C_{n+1} \to (0,\infty)$ be a surjection such that $f(x)/x \notin \mathbb{Q}$ for every $x \in C_{n+1}$. After this construction is completed, for the remaining set $(0,\infty) \setminus \bigcup_{n\in\mathbb{N}} C_n$ put $f(x) = \sqrt{2}$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise.

Clearly, the function $f\colon (0,\infty) \to (0,\infty)$ is Darboux, because $f([a,b]) = (0,\infty)$ for any $0 < a < b$. It remains to show that its graph does not contain any connected sets with more than one point. Let E be an arbitrary subset of the graph containing two distinct points. For some $(a, f(a)) \in E$ let $\alpha = f(a)/a$. If E is contained in the line $y = \alpha x$, it cannot be connected because then f would be continuous on some interval. Otherwise there is some $(b, f(b)) \in E$ with $\beta = f(b)/b$, $\alpha \neq \beta$. Choose a rational number γ between α and β. Now, the line $y = \gamma x$ separates E. So we showed that the graph of f is totally disconnected. □

Now that we are sensitive to the difference between a connectivity function and a Darboux function, let us consider an alternative version of Theorem 2, where the connectivity function is replaced with a Darboux function at the cost of requiring Y to be a T_3 space.

Theorem 7. *Let Y be a T_3 space. Let $F\colon \mathbb{R} \to Y$ be a Darboux function. If F is locally connected at $(x_0, F(x_0))$, then F is continuous at x_0.*

Proof. Take any open set $U \subset Y$ containing $F(x_0)$. Since F is locally connected at $(x_0, F(x_0))$, there is a connected set $K \subset F$ such that $(x_0, F(x_0)) \in Int_F(K) \subset K \subset \mathbb{R} \times U$. Hence, since Y is T_3, there is an open set $V \subset Y$ such that $F(x_0) \in V \subset \overline{V} \subset U$ and a radius $r > 0$ such that $F \cap ((x_0 - r, x_0 + r) \times \overline{V}) \subset K$. Suppose that $(x, F(x)) \notin K$ for all $x \in (x_0, x_0 + r)$. Then $F((x_0, x_0 + r)) \subset Y \setminus \overline{V}$. Since F is a Darboux function, the set $A = F([x_0, x_0 + r))$ is connected and covered by the union of two disjoint open sets $V \cup (Y \setminus \overline{V})$, each of them intersecting A. This contradiction shows that there is a point $b \in (x_0, x_0 + r)$ such that $b \in dom(K)$. Since the projection $dom(K)$ is an interval, it contains $[x_0, b]$. Similarly, there exists a point $a \in (x_0 - r, x_0)$ such that $[a, x_0] \subset dom(K)$. So, $x_0 \in (a, b) \subset dom(K)$ and $F((a, b)) \subset U$, which finishes the argument that F is continuous at x_0. □

2.2 Connected Graph with a Disconnected Complement

The main theorem in this section relies on the following folklore lemma, which deserves much interest for its own sake. It characterizes continuity by saying that $f\colon X \to \mathbb{R}$ is continuous if and only if the set of points above the graph is open and the set of points below the graph is open.

Lemma 8. *Let X be a topological space and $f\colon X \to \mathbb{R}$. Then f is upper semicontinuous if and only if the set $A = \{(x, y)\colon f(x) < y\}$ is open in $X \times \mathbb{R}$. Similarly, f is lower semicontinuous if and only if the set $B = \{(x, y)\colon f(x) > y\}$ is open in $X \times \mathbb{R}$. Consequently, f is continuous if and only if the sets A and B are open.*

The proof of this lemma is elementary. The main theorem is given below.

Theorem 9. *If X is a topological space, $f\colon X \to \mathbb{R}$, $Gr(f)$ is connected, and $(X \times \mathbb{R}) \setminus Gr(f)$ is disconnected then f is continuous.*

Proof. Since $(X \times \mathbb{R}) \setminus Gr(f)$ is disconnected, there exist two open sets $A, B \subset X \times \mathbb{R}$ such that $(X \times \mathbb{R}) \setminus Gr(f) \subset A \cup B$, $A \cap B \setminus Gr(f) = \emptyset$, $A \setminus Gr(f) \neq \emptyset$, and $B \setminus Gr(f) \neq \emptyset$. Notice that $A \cap B \subset Gr(f)$. Hence $A \cap B \subset Int(Gr(f)) = \emptyset$. Furthermore, since A, B are disjoint open sets, we have that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Using this and $(X \times \mathbb{R}) \setminus Gr(f) \subset A \cup B$ we get that $\overline{A} \setminus Gr(f) \subset A$ and $\overline{B} \setminus Gr(f) \subset B$. Since the sets $\{x\} \times (f(x), \infty)$ and $\{x\} \times (-\infty, f(x))$ are connected and contained in the complement of the graph and A, B are disjoint open sets whose union covers the complement of the graph, we have the following four easy consequences:

(i) $(x, f(x) + 1) \in A \implies \{x\} \times (f(x), \infty) \subset A$

(ii) $(x, f(x) + 1) \in B \implies \{x\} \times (f(x), \infty) \subset B$

(iii) $(x, f(x) - 1) \in A \implies \{x\} \times (-\infty, f(x)) \subset A$

(iv) $(x, f(x) - 1) \in B \implies \{x\} \times (-\infty, f(x)) \subset B$.

For $K, L \in \{A, B\}$, let $G^K_L = \{(x, f(x))\colon (x, f(x) + 1) \in K \wedge (x, f(x) - 1) \in L\}$. Notice that since A, B are disjoint, the sets $G^A_A, G^A_B, G^B_A, G^B_B$ are pairwise disjoint, and since A and B cover the complement of the graph, we have that $Gr(f) = G^A_A \cup G^A_B \cup G^B_A \cup G^B_B$. We are going to show that these four sets are closed subsets of the graph, which — since the graph is connected — implies that three of them are empty and one of them is the whole graph. Take any net $(x_t, f(x_t))$ contained in G^K_L and converging to $(x, f(x))$. Since $(x_t, f(x_t) + 1) \in K$, $(x, f(x) + 1) \in \overline{K} \setminus Gr(f) \subset K$, and similarly since $(x_t, f(x_t) - 1) \in L$, $(x, f(x) - 1) \in \overline{L} \setminus Gr(f) \subset L$. Hence $(x, f(x)) \in G^K_L$, which shows that the set G^K_L is closed. Knowing that the whole graph is equal to one of these four sets and recalling that $A \setminus Gr(f) \neq \emptyset$ and $B \setminus Gr(f) \neq \emptyset$, we conclude that either $Gr(f) = G^A_B$ or $Gr(f) = G^B_A$. Without loss of generality we may assume that $Gr(f) = G^A_B$. It is now easy to notice that

$$\{(x, y)\colon f(x) < y\} \subset A,$$

$$\{(x, y)\colon f(x) > y\} \subset B.$$

Next, we are going to show that $A \cap Gr(f) = \emptyset$ and $B \cap Gr(f) = \emptyset$. Suppose we have $(x, f(x)) \in A \cap Gr(f)$. Since A is open, there exist two open sets $U \subset X$ and $V \subset \mathbb{R}$ such that $(x, f(x)) \in U \times V \subset A$. Hence there exists an $\varepsilon > 0$ such that $(x, f(x) - \varepsilon) \in A$ but $(x, f(x) - \varepsilon) \in B$ and $A \cap B = \emptyset$. This contradiction shows that $A \cap Gr(f) = \emptyset$ and analogously $B \cap Gr(f) = \emptyset$. Thus we have

$$\{(x, y)\colon f(x) < y\} = A,$$

$$\{(x, y)\colon f(x) > y\} = B,$$

and since A, B are open, Lemma 8 implies that our function is continuous. □

Corollary 10. *Let X be a connected space and $f\colon X \to \mathbb{R}$. Then f is continuous if and only if $Gr(f)$ is connected and $(X \times \mathbb{R}) \setminus Gr(f)$ is disconnected.*

Proof. If $Gr(f)$ is connected and $(X \times \mathbb{R}) \setminus Gr(f)$ is disconnected, then f is continuous by Theorem 9. Now, if f is continuous, then $Gr(f)$ is homeomorphic with X via $\Theta(x, f(x)) = x$. Hence, since X is connected, $Gr(f)$ is connected. Using Lemma 8 we show that $A = \{(x,y)\colon f(x) < y\}$ and $B = \{(x,y)\colon f(x) > y\}$ are disjoint open sets covering the complement of the graph, hence $(X \times \mathbb{R}) \setminus Gr(f)$ is disconnected. □

It is essential that $\mathbb{R}$ should be the range of the function for this characterization of continuity because the fact that the complement of the graph of a continuous real-valued function is disconnected follows from the following property of the real line — it becomes disconnected if you take away one point. It makes no sense to replace $\mathbb{R}$ with a multi-dimensional vector space because for any space Y such that it is still connected after removing any of its points the complement of the graph of any function $f\colon X \to Y$ (not necessarily continuous) is connected.

Theorem 11. *If X is a connected space, Y is a topological space such that for every $y \in Y$ the set $Y \setminus \{y\}$ is connected and has at least two elements, and $f\colon X \to Y$ is an arbitrary function, then $(X \times Y) \setminus Gr(f)$ is connected.*

Proof. Take any open sets $A, B \subset X \times Y$ such that $(X \times Y) \setminus Gr(f) \subset A \cup B$, $A \cap B \setminus Gr(f) = \emptyset$, and $A \setminus Gr(f) \neq \emptyset$. Now, if we show that $B \setminus Gr(f) = \emptyset$, we have proved that $(X \times Y) \setminus Gr(f)$ is connected.

Using the fact that the sets $\{x\} \times (Y \setminus \{f(x)\})$ are connected we obtain the following two easy consequences:

(i) $(x,y) \in A \setminus Gr(f) \implies \{x\} \times (Y \setminus \{f(x)\}) \subset A$

(ii) $(x,y) \in B \setminus Gr(f) \implies \{x\} \times (Y \setminus \{f(x)\}) \subset B.$

Let $E = \{x \in X\colon \{x\} \times (Y \setminus \{f(x)\}) \subset A\}$. Recalling that $A \setminus Gr(f) \neq \emptyset$ and keeping (i) in mind, we conclude that E is not empty. We are going to show that E is closed. Suppose that E is not closed. Then there exists a point $x \in \overline{E} \setminus E$. Since $x \notin E$ — keeping (ii) in mind — we conclude that $\{x\} \times (Y \setminus \{f(x)\}) \subset B$. There exist two distinct points $y_1, y_2 \in Y$ such that $f(x) \neq y_1$ and $f(x) \neq y_2$. Notice that $(x, y_1) \in B$ and $(x, y_2) \in B$. Since B is open, there exist open sets $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$ such that $(x, y_1) \in U_1 \times V_1 \subset B$ and $(x, y_2) \in U_2 \times V_2 \subset B$. Since $x \in U_1 \cap U_2 \cap \overline{E}$, there exists a point $a \in U_1 \cap U_2 \cap E$. Since f is a function, either $f(a) \neq y_1$ or $f(a) \neq y_2$. So, $f(a) \neq y_i$ for some $i \in \{1,2\}$. Since $a \in E$, $(a, y_i) \in A$. But $(a, y_i) \in U_i \times V_i \subset B$. So (a, y_i) belongs to the empty set $A \cap B \setminus Gr(f)$. This contradiction shows that E is closed.

Similarly, we show that $F = \{x \in X\colon \{x\} \times (Y \setminus \{f(x)\}) \subset B\}$ is closed. Then $X = E \cup F$ while E, F are closed and disjoint. Since X is connected and E is not empty, F must be empty. This shows that $(X \times Y) \setminus Gr(f) \subset A$. Hence $B \setminus Gr(f) = \emptyset$, which concludes the proof. □

Chapter 3

Functions with Closed Connected Graphs

In this chapter we prove that a function $f\colon \mathbb{R} \to \mathbb{R}^n$ is continuous if and only if its graph is closed and connected and give an example of a discontinuous function $f\colon \mathbb{R} \to l^2$ with a closed connected graph. At this point it is interesting to ask whether a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with a closed connected graph has to be continuous. We give a positive answer in the very special case when connectedness of the graph follows from the assumption that all x-sections are continuous and one y-section is continuous. A negative answer was given by Jelinek — see Appendix A.

The following three theorems are known folklore results showing the basic properties of functions with closed graphs into spaces with appropriate compactness properties.

Theorem 12. *If X, Y are topological spaces, $f\colon X \to Y$, $Gr(f)$ is closed, $E \subset Y$, and E is compact, then $f^{-1}(E)$ is closed.*

Proof. Take any $x_0 \in \overline{f^{-1}(E)}$. We have a net $(x_t)_{t\in\Pi}$ in $f^{-1}(E)$ which converges to x_0. Notice that $(f(x_t))_{t\in\Pi}$ is a net in the compact set E. Hence we get a subnet $(f(x_{\alpha(s)}))_{s\in\Pi_0}$ which converges to some $y_0 \in E$. Notice that $(x_{\alpha(s)}, f(x_{\alpha(s)})) \to (x_0, y_0)$. Since $Gr(f)$ is closed, $f(x_0) = y_0$. So $f(x_0) \in E$, and $x_0 \in f^{-1}(E)$. We showed that $\overline{f^{-1}(E)} \subset f^{-1}(E)$, so $f^{-1}(E)$ is closed. □

Theorem 13. *If X is a topological space, Y is a compact space, $f\colon X \to Y$, and $Gr(f)$ is closed, then f is continuous.*

Proof. Take any closed set $E \subset Y$. Since Y is a compact space, E is compact. By Theorem 12, $f^{-1}(E)$ is closed. We showed that f is continuous. □

Theorem 14. *If X is a topological space, Y is a locally compact space, $f\colon X \to Y$, and $Gr(f)$ is closed, then $W = \{x \in X : f \text{ is continuous at } x\}$ is open.*

Proof. Take any $x_0 \in W$. Since Y is locally compact, we have an open set $U \subset Y$ such that $f(x_0) \in U$ and $\overline{U}$ is compact. Since f is continuous at x_0, we have an open $G \subset X$ such that $x_0 \in G$ and $f(G) \subset U$. Notice that $f|_G \colon G \to \overline{U}$ and $Gr(f|_G)$ is closed in $G \times \overline{U}$. By Theorem 13, $f|_G$ is continuous. Since G is open, f is continuous on G. Hence $G \subset W$, and so $x_0 \in Int(W)$. We showed that $W \subset Int(W)$, so W is open. $\square$

The following technical lemma is so far the most general and efficient tool for deriving the continuity of a function with a closed graph coupled with an appropriate connectedness condition.

Lemma 15. *Let X be a topological space. Let Y be a locally compact space. Let $x_0 \in A \subset X$. Suppose that $f \colon X \to Y$ has a closed graph and $f|_A$ is continuous at x_0. Suppose that for every neighborhood G of x_0 there is a smaller neighborhood G' of x_0 such that for every $z \in G'$ there is a set $E \subset G$ containing z such that $E \cap A \neq \emptyset$ and $f(E)$ is connected. Then f is continuous at x_0.*

Proof. Since Y is locally compact at $f(x_0)$, there is an open set $U_0 \subset Y$ such that $f(x_0) \in U_0$ and $\overline{U_0}$ is compact. Take any open set U such that $f(x_0) \in U \subset U_0$. Since f has a closed graph and $\overline{U} \setminus U$ is compact, by Theorem 12, there is a neighborhood G_1 of x_0 such that $f(G_1) \cap (\overline{U} \setminus U) = \emptyset$. Since $f|_A$ is continuous at x_0, there is a neighborhood G_2 of x_0 such that $f(G_2 \cap A) \subset U$. Let $G = G_1 \cap G_2$. Then there is a neighborhood $G' \subset G$ of x_0 with the properties postulated in the last assumption. Take any $z \in G'$. Then $z \in E \subset G$, $a \in E \cap A$ and $f(E)$ is connected. Since $E \subset G_1$, we can write $f(E) \subset U \cup (Y \setminus \overline{U})$, which is a covering of a connected set by two disjoint open sets. Since $a \in A \cap E \subset A \cap G \subset A \cap G_2$, $f(a) \in f(E) \cap U$. So $f(E) \subset U$ and consequently $f(z) \in U$. Thus we showed that $f(G') \subset U$, which completes the proof that f is continuous at x_0. $\square$

A straightforward application of the previous highly technical lemma yields a series of corollaries.

Corollary 16. *If X is a topological space, Y is a locally connected space, Z is a locally compact space, $f \colon X \times Y \to Z$, $Gr(f)$ is closed, $y_0 \in Y$,*
(1) the mapping $Y \ni y \mapsto f(x, y) \in Z$ is Darboux for all $x \in X$,
(2) the mapping $X \ni x \mapsto f(x, y_0) \in Z$ is continuous,
then f is continuous at (x_0, y_0) for all $x_0 \in X$.

Proof. Take any $x_0 \in X$. We are preparing to apply Lemma 15. Let $A = X \times \{y_0\}$. By (2), $f|_A$ is continuous at (x_0, y_0). Take any open sets U and V such that $(x_0, y_0) \in U \times V$. Since Y is locally connected, there is a connected neighborhood K of y_0 with $K \subset V$. Let $G' = U \times Int(K)$. Take any $v = (x, y) \in G'$. Let $E = \{x\} \times K$. By (1), $f(E)$ is connected. Notice that $v \in E$ and $(x, y_0) \in E \cap A$. Now, ready to apply Lemma 15, we conclude that f is continuous at (x_0, y_0). Since $x_0 \in X$ was arbitrary, the proof is complete. $\square$

The following corollary is a previously known result [9].

Corollary 17. *If X is a topological space, Y is a locally connected space, Z is a locally compact space, $f\colon X \times Y \to Z$ has a closed graph,*
(1) the mapping $Y \ni y \mapsto f(x, y) \in Z$ is Darboux for every $x \in X$,
(2) the mapping $X \ni x \mapsto f(x, y) \in Z$ is continuous for every $y \in Y$,
then f is continuous.

By taking a singleton for X in the previous corollary, we obtain the simplest theorem about a Darboux function with a closed graph.

Corollary 18. *If Y is a locally connected space, Z is a locally compact space, $f\colon Y \to Z$ is a Darboux function with a closed graph, then f is continuous.*

Calling to mind the fact that a function $f\colon \mathbb{R} \to Y$ with a connected graph has to be Darboux, we obtain the previously announced characterization of continuity in terms of properties of the graph.

Corollary 19. *If Y is a locally compact space and $f\colon \mathbb{R} \to Y$ has a closed connected graph then f is continuous.*

The following example shows that it is essential that Y should be locally compact in Corollary 18.

Example 20. *There exists a discontinuous function $f\colon [0, 1] \to l^2$ with a closed connected graph.*

Proof. Let $Y = l^2$. Let $(x_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers such that

$$\{x_n\colon n \in \mathbb{N}\} = \{x \in (0, 1]\colon \sin(\frac{\pi}{x}) = 0\}.$$

It follows that $x_1 = 1$. Let us define our function $f\colon [0, 1] \to Y$. Let $f(0) = (0, 0, 0, ...) \in Y$. Take any $x \in (0, 1] = (0, x_1]$. There is a unique $n \in \mathbb{N}$ such that $x_{n+1} < x \leq x_n$. For each $k \in \mathbb{N}$ define

$$f(x)(k) = \begin{cases} \sin(\frac{\pi}{x}) & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

Since $f(x)$ is an element of Y, our function f has been defined. It is easy to see that $f|_{[x_{n+1},x_n]}$ is continuous for every $n \in \mathbb{N}$. Hence $f|_{[x_{n+1},1]}$ is continuous for every $n \in \mathbb{N}$, which means that f is continuous for every $x \in (0, 1]$. Since $(x_n, f(x_n)) = (x_n, (0, 0, 0, ...)) = (x_n, f(0)) \to (0, f(0))$ as $n \to \infty$, we conclude that f has a connected graph.

Let $(a_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers such that

$$\{a_n\colon n \in \mathbb{N}\} = \{x \in (0, 1]\colon \sin(\frac{\pi}{x}) = 1\}.$$

Take any $n \in \mathbb{N}$. Choose $k \in \mathbb{N}$ so that $x_{k+1} < a_n < x_k$. Thus $f(a_n)(k) = \sin(\frac{\pi}{a_n}) = 1$ and so $\|f(a_n)\| = 1$ for every $n \in \mathbb{N}$, which means that f is discontinuous at 0.

It remains to show that f has a closed graph. Take any sequence $(z_n)_{n\in\mathbb{N}}$ converging to 0 such that $f(z_n)_{n\in\mathbb{N}}$ converges to some point $y \in Y$. We will show that $y = (0,0,0,...) = f(0)$. Take any $k \in \mathbb{N}$. Take any $\varepsilon > 0$. Since $\|f(z_n) - y\| \to 0$ as $n \to \infty$, there is an $n_0 \in \mathbb{N}$ such that

$$|f(z_n)(k) - y(k)| \leq \varepsilon$$

for every $n \geq n_0$. Choose an $n \in \mathbb{N}$ so that $n > n_0$ and $z_n < x_{k+1}$. Then $f(z_n)(k) = 0$. So $|y(k)| \leq \varepsilon$ for every $\varepsilon > 0$, and consequently $y(k) = 0$ for every $k \in \mathbb{N}$. This shows that the graph of f is closed. $\square$

Piotrowski and Wingler noticed that a separately continuous function $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with a closed graph has to be continuous — see [9]. The exact statement of their theorem is our Corollary 17. However, the assumption of separate continuity can be considerably weakened and replaced by requiring that all x-sections are continuous and at least one y-section is continuous. We have even two ways of proving this in a general topological setting. In both cases we assume that we have a function $f\colon X \times Y \to Z$ with a closed graph into a locally compact space Z whose all x-sections are continuous and at least one y-sections is continuous, with Y being connected and locally connected. In the first theorem we require additionally that X be locally connected and in the second theorem we require nothing of X while demanding local compactness from Y.

We can easily see that the condition *Z is locally compact* is essential by taking $Z = \mathbb{R} \times \mathbb{R}$ with the river metric and $f\colon \mathbb{R}\times\mathbb{R} \to Z$ simply given by $f(x, y) = (x, y) \in Z$. Then all x-sections are continuous and exactly one y-section is continuous and the graph is closed because the inverse function is continuous. Moreover, the set of continuity points is $\mathbb{R} \times \{0\}$, which is not open, showing that local compactness is essential in Theorem 14.

Theorem 21. *If X is a locally connected space, Y is a connected and locally connected space, Z is a locally compact space, $f\colon X \times Y \to Z$, $Gr(f)$ is closed,*
(1) the mapping $Y \ni y \mapsto f(x, y) \in Z$ is continuous for all $x \in X$,
(2) the mapping $X \ni x \mapsto f(x, y) \in Z$ is continuous for some $y \in Y$
then f is continuous.

Proof. Let $W = \{(x, y) \in X \times Y : f \text{ is continuous at } (x, y)\}$. By Theorem 14, W is open. Take any $x_0 \in X$. Let $D = \{y \in Y : f \text{ is continuous at } (x_0, y)\}$. Notice that D is open in Y because W is open in $X \times Y$. We will show that D is closed in Y. Take any $y_0 \in \overline{D}$. Let $\Pi = \{U \subset X \times Y : (x_0, y_0) \in U$ and U is open$\}$. Let $P = \{E \subset X \times Y : f(E) \text{ is connected}\}$. Let $A = \{(x_0, y) : y \in Y\}$. Take any $U \in \Pi$. We have open sets $G_X \subset X$, $G_Y \subset Y$ such that $(x_0, y_0) \in G_X \times G_Y \subset U$. Since Y is locally connected, we have a connected set $K \subset G_Y$ such that $y_0 \in Int(K)$. Since $y_0 \in \overline{D}$, we can choose a $y' \in Int(K) \cap D$. Since $y' \in D$, $(x_0, y') \in W$. Since W is open, we have an open set $V \subset X \times Y$ such that $(x_0, y') \in V$ and f is continuous on V. Now, we have open sets $V_X \subset X$, $V_Y \subset Y$ such that $(x_0, y') \in V_X \times V_Y \subset V \cap (G_X \times Int(K))$. Since X is locally connected, we have a connected set $T_X \subset V_X$ such that $x_0 \in Int(T_X)$. Since Y is locally connected, we have a connected set $T_Y \subset V_Y$ such that $y' \in Int(T_Y)$. Let $G = Int(T_X) \times Int(K)$. $G \in \Pi$ because $x_0 \in Int(T_X)$ and $y_0 \in Int(K)$. Take any $g = (v, z) \in G$. Let $E = T_X \times T_Y \cup \{v\} \times K$. We will show that $E \in P$. Notice that $T_Y \subset K$ and $v \in T_X$. So $T_X \times T_Y \cap \{v\} \times K \neq \emptyset$. Hence $f(T_X \times T_Y) \cap f(\{v\} \times K) \neq \emptyset$. Now, $T_X \times T_Y$ is connected and contained in V. Since f is continuous on V, $f(T_X \times T_Y)$ is connected. By (1), $f(\{v\} \times K)$ is connected. Notice that $f(E) = f(T_X \times T_Y) \cup f(\{v\} \times K)$. Hence $f(E)$ is connected. So $E \in P$. Notice that $(v, z) \in E$. Notice that $E \subset U$. We have $(x_0, y') \in T_X \times T_Y \subset E$ and $(x_0, y') \in A$. So $E \cap A \neq \emptyset$. We showed that

$$\forall_{U \in \Pi} \exists_{G \in \Pi} \forall_{g \in G} \exists_{E \in P}\, g \in E \wedge E \subset U \wedge E \cap A \neq \emptyset.$$

By (1), $f|_A$ is continuous at (x_0, y_0). By Lemma 15, f is continuous at (x_0, y_0). So $y_0 \in D$. We showed that $\overline{D} \subset D$. So D is closed in Y. So D is open and closed in Y. By (2), we have a $y \in Y$ such that the mapping $X \ni x \mapsto f(x, y) \in Z$ is continuous. By Corollary 16, we conclude that f is continuous at (x_0, y). So $y \in D$ and $D \neq \emptyset$. Since Y is connected, $D = Y$. Hence f is continuous at (x_0, y) for all $y \in Y$. But $x_0 \in X$ was arbitrary. Thus f is continuous, and the proof is complete. □

Theorem 22. *If X is a topological space, Y is a connected, locally connected, locally compact space, Z is a locally compact space, $f\colon X \times Y \to Z$ has a closed graph,*
(1) the mapping $Y \ni y \mapsto f(x_0, y) \in Z$ is Darboux for each $x_0 \in X$,
(2) the mapping $X \ni x \mapsto f(x, y_1) \in Z$ is continuous for some $y_1 \in Y$,
then f is continuous.

Proof. By Corollary 18, we can strengthen (1) so that the mapping $Y \ni y \mapsto f(x_0, y) \in Z$ is continuous for each $x_0 \in X$, rather than Darboux, which will be used to map compact subsets of Y onto compact subsets of Z.

Take any $x_0 \in X$. Let $D = \{y \in Y : f \text{ is continuous at } (x_0, y)\}$. By Theorem 14, D is open and by Corollary 16, $y_1 \in D$. Since Y is connected, we are done as soon as we show that D is closed. Let us take $y_0 \in \overline{D} \setminus D$ and the proof will be finished as soon as we reach a contradiction. Since Y is locally connected and locally compact, there is a connected and compact neighborhood K of y_0. Let $E = f(\{x_0\} \times K)$. By the strengthened version of (1), E is compact. Since Z is locally compact, we can obtain a closed compact set M such that $E \subset Int(M)$. Since $y_0 \in \overline{D}$, there is a point $y' \in D \cap Int(K)$. Now, (x_0, y') is a point of continuity with $f(x_0, y') \in Int(M)$, so there is a neighborhood G of x_0 such that $f(x, y') \in Int(M)$ for every $x \in G$. Since (x_0, y_0) is a point of discontinuity, due to Theorem 13, there is no neighborhood U of (x_0, y_0) with $f(U) \subset M$. So there is a net (x_t, y_t) converging to (x_0, y_0) such that $x_t \in G$, $y_t \in Int(K)$ and $f(x_t, y_t) \in Z \setminus M$, and consequently $f(x_t, y') \in Int(M)$. By (1), the sets $f(\{x_t\} \times K)$ are connected. Since each of them intersects both $Z \setminus M$ and $Int(M)$, there are points $y'_t \in K$ such that $f(x_t, y'_t) \in M \setminus Int(M)$. Since K is compact, there is a subnet $y'_{\alpha(s)}$ converging to some point $y'' \in K$.

Now, the net $z_s = f(x_{\alpha(s)}, y'_{\alpha(s)})$ is contained in the compact set $M \setminus Int(M)$, and so there is another subnet $z_{\beta(w)}$ converging to some point $z_0 \in M \setminus Int(M)$. Since the graph of f is closed, $f(x_0, y'') = z_0 \in f(\{x_0\} \times K) = E \subset Int(M)$, showing that $z_0 \in Int(M)$, which contradicts with $z_0 \in M \setminus Int(M)$. $\square$

Chapter 4

Punctiform Spaces as Connected Graphs of Functions

A topological space is called *separably connected* if any two of its points can be contained in a connected separable subset. Clearly, this is a generalization of path connectedness. This concept arises naturally in the course of investigating the question whether a continuous real-valued function from a connected space having a local extremum everywhere has to be constant. A partial positive answer is given for separably connected metric spaces.

There are easy and natural examples of nonmetrizable connected spaces which are not separably connected. However, connected *metric* spaces, which are not separably connected, are hard to find. There are four examples of such spaces in the literature — see [10], [11], [2] — and we have constructed one more example by using a method which turns out to have much broader applications. In fact, we discovered a certain way of constructing functions with connected dense graphs inside a broad class of product spaces.

Our construction is based on two ideas. Firstly, it is inspired by Bernstein's Connected Sets — see [12] for details. Secondly, our space is defined as a function from the real line whose graph is a connected dense subset of a nonseparable metric space. Consequently, our space does not contain any nontrivial connected separable subsets and is not locally connected at any point.

It is still an open question whether there exists a *complete* connected metric space which fails to be separably connected. However, we obtained a *complete connected punctiform metric space* as a closed connected graph of a function from the real line into a complete metric space. (A punctiform space contains no nontrivial compact connected subsets.)

4.1 Functions with Connected Dense Graphs

Our method for producing a connected, not separably connected metric space is in fact a tool for constructing a function $f\colon X \to Y$ with a connected dense graph for a broad class of products $X \times Y$, including any normed spaces X and Y of size $\mathfrak{c}$, in which case such a function may be required to satisfy Cauchy's equation $f(x+u) = f(x) + f(u)$. See Kulpa's paper [5] for other results of this kind. There is an old classical construction due to Leopold Vietoris of a function $f\colon [0,1] \to [0,1]$ with a connected dense graph, [13].

The following technical lemmas are painstakingly written so as to first of all reveal all the details that are essential for our construction of a connected, not separably connected metric space, and secondly get rid of all the details which have no significance for the construction.

Lemma 23. *Let X, Y and $\mathcal{H} \subset \mathcal{P}(X \times Y)$ be arbitrary sets such that $|dom(K)| \geq |\mathcal{H}|$ for every $K \in \mathcal{H}$. Then there exists a function $f\colon X \to Y$ which intersects every member of the family $\mathcal{H}$.*

Proof. Let the ordinal $\Gamma = card(\mathcal{H})$ be used to well-order the family $\mathcal{H} = \{K_\alpha\colon \alpha < \Gamma\}$. We will be constructing the desired function by producing one by one elements of its graph $(x_\alpha, f(x_\alpha))$ by transfinite induction over $\alpha < \Gamma$. In the first step, pick some $(x_0, f(x_0)) \in K_0$. Now, given an ordinal $\beta < \Gamma$, notice that the set $dom(K_\beta) \setminus \{x_\alpha\colon \alpha < \beta\}$ is not empty because of our cardinality constraint. So we can safely choose some x_β and some corresponding $f(x_\beta)$ with $(x_\beta, f(x_\beta)) \in K_\beta$. Should the set $X \setminus \{x_\alpha\colon \alpha < \Gamma\}$ remain nonempty, fill the graph of our function in an arbitrary way just to extend the domain to the whole X. $\square$

It is worth noting that in the following lemma Y is assumed to be separably connected, which is a nice advertisement for this little known topological property.

Lemma 24. *Let X be a connected space of size κ whose each nonempty open subset contains a closed separable set of size κ. Let Y be a separably connected T_1 space. Suppose that $X \times Y$ is a T_5 space whose separable subsets are hereditarily separable. Let $\mathcal{H}$ be the family of all closed separable subsets of $X \times Y$ whose projection on X has size κ. Let the function $F\colon X \to Y$ intersect every member of $\mathcal{H}$. Then the graph of F is connected and dense in $X \times Y$.*

Proof. To see that F is dense notice that it intersects every set of the form $E \times \{y\}$ — where E is a closed separable set of size κ — which is a set in $\mathcal{H}$ to be found in every nonempty open subset of $X \times Y$.

Suppose that F is disconnected. Since $X \times Y$ is T_5, there are two disjoint open sets $U, V \subset X \times Y$ such that $F \subset U \cup V$, $F \cap U \neq \emptyset$, $F \cap V \neq \emptyset$. Let $A = dom(U)$ and $B = dom(V)$. These sets, as projections of open sets, are open in X. Since F is a function with domain X, for every $x \in X$ there is a $y \in Y$ with $(x, y) \in F \subset U \cup V$, and thus $X = A \cup B$. Since $F \cap U \neq \emptyset$, A

is nonempty. Similarly, B is nonempty. Since X is connected it follows that $A \cap B$ is nonempty. Hence there is a point $x_0 \in X$ and two points $y_1, y_2 \in Y$ with $(x_0, y_1) \in U$ and $(x_0, y_2) \in V$. Since U, V are open, there is an open set G containing x_0 such that $G \times \{y_1\} \subset U$ and $G \times \{y_2\} \subset V$. By assumption, there is a closed separable set $E \subset G$ of size κ. Since Y is separably connected, there is a closed connected separable set $W \subset Y$ containing both y_1 and y_2. Notice that the set $K = (E \times W) \setminus (U \cup V)$ is closed and separable in $X \times Y$. To show that $E \subset dom(K)$ take any $x \in E$. Since $(x, y_1) \in U$ and $(x, y_2) \in V$, the connected set $\{x\} \times W$ intersects both U and V, which are disjoint open sets, hence it cannot be covered by their union. Thus $(x, y) \in (\{x\} \times W) \setminus (U \cup V) \subset K$ for some $y \in Y$ ensuring that $x \in dom(K)$. We showed that $dom(K)$ contains the set E of size κ. Thus $K \in \mathcal{H}$ and by assumption $F \cap K \neq \emptyset$. But $F \cap K = \emptyset$. This contradiction shows that F is connected. □

Recall that every separable metric space can be embedded in the Hilbert cube $[0,1]^{\mathbb{N}}$ with the product metric, which has size $\mathfrak{c}$. It is interesting to realize that there is also a similar constraint on the cardinality of Hausdorff separable spaces, which cannot exceed $2^{\mathfrak{c}}$ — see [12] at the end of section *Compactness Properties and the T_i axioms*.

Lemma 25. *Let Z be an arbitrary topological space. Let $\mathcal{H}$ be the family of all closed separable subsets of Z. Then $|\mathcal{H}| \leq |Z|^{\aleph_0}$.*

Proof. Let A, B be two distinct closed separable subsets of Z. Then there are two sequences $a, b \in Z^{\mathbb{N}}$ such that $\overline{a(\mathbb{N})} = A$ and $\overline{b(\mathbb{N})} = B$. Since $A \neq B$, it follows that $a \neq b$. This means that to each closed separable set we can assign a unique element of $Z^{\mathbb{N}}$, which completes the argument. □

The following theorem yields the existence of functions $F\colon X \to Y$ with connected dense graphs inside a broad class of products $X \times Y$.

Theorem 26. *Let X be a connected space whose each nonempty open subset contains a closed separable set of size $\mathfrak{c}$. Let Y be a separably connected space. Suppose that $X \times Y$ is a T_5 space of size $\mathfrak{c}$ whose separable subsets are hereditarily separable. Then there exists a function $F\colon X \to Y$ whose graph is connected and dense in $X \times Y$.*

Proof. Let $\mathcal{H}$ be the family of all closed and separable subsets of $X \times Y$ whose projection on X has size $\mathfrak{c}$. By Lemma 25, $|\mathcal{H}| \leq \mathfrak{c}^{\aleph_0} = \mathfrak{c}$. Thus $X, Y, \mathcal{H}$ satisfy the assumptions of Lemma 23. So there exists a function $F\colon X \to Y$ which intersects every member of $\mathcal{H}$. By Lemma 24, the graph of F is connected and dense in $X \times Y$. □

In the next theorem we show that if X, Y are normed spaces, F may satisfy Cauchy's equation.

Theorem 27. *Let X, Y be normed spaces of size $\mathfrak{c}$. Then there exists a function $F\colon X \to Y$ satisfying $F(x+u) = F(x) + F(u)$ for all $x, u \in X$ whose graph is connected and dense in $X \times Y$.*

Proof. Let $\mathcal{H}$ be the family of all closed and separable subsets of $X \times Y$ whose projection on X has size $\mathfrak{c}$. By Lemma 25, $|\mathcal{H}| \leq \mathfrak{c}^{\aleph_0} = \mathfrak{c}$. If $E \subset X$ then let $\mathrm{span}_{\mathbb{Q}}(E) = \{\Sigma_{i=1}^n a_i x_i \colon a_i \in \mathbb{Q},\ x_i \in E,\ i \in [1,n] \cap \mathbb{N},\ n \in \mathbb{N}\}$. Notice that $|E| < \mathfrak{c} \implies |\mathrm{span}_{\mathbb{Q}}(E)| < \mathfrak{c}$. Let us write $\mathcal{H} = \{K_\alpha \colon \alpha < \mathfrak{c}\}$. By transfinite induction over $\alpha < \mathfrak{c}$ we will construct a linearly independent subset B_0 of vectors in X over the field of rational numbers and a function $F \colon B_0 \to Y$ intersecting every member of $\mathcal{H}$. In the step zero, we choose a nonzero vector x_0 such that $(x_0, F(x_0)) \in K_0$. In the βth step, we choose an $x_\beta \in dom(K_\beta) \setminus \mathrm{span}_{\mathbb{Q}}(\{x_\alpha \colon \alpha < \beta\})$ and some $F(x_\beta)$ such that $(x_\beta, F(x_\beta)) \in K_\beta$. Let $B_0 = \{x_\alpha \colon \alpha < \mathfrak{c}\}$. Let us extend F to the whole X in the following way. Let B be a Hamel basis for X over the field of rational numbers containing the linearly independent set B_0. Put $F(x) = 0$ for all $x \in B \setminus B_0$ and put $F(\Sigma_{i=1}^n a_i x_i) = \Sigma_{i=1}^n a_i F(x_i)$ for all $a_i \in \mathbb{Q},\ x_i \in B,\ i \in [1,n] \cap \mathbb{N},\ n \in \mathbb{N}$. Now, the function F satisfies $F(x+u) = F(x) + F(u)$ and by Lemma 24 its graph is connected and dense in $X \times Y$. □

4.2 A Nonseparably Connected Metric Space

We are now ready to show our example of a connected, not separably connected metric space. It will soon be clear why we had to choose the real line for the domain of our function with a connected dense graph to ensure that it does not contain any nontrivial connected separable subsets and to conclude that it is not locally connected at any point.

Theorem 28. *There exists a nonseparable connected metric space* M *of size* $\mathfrak{c}$ *with the following properties:*

(1) each separable connected subset of M *is a singleton,*

(2) $\mathrm{M} \setminus \{p\}$ *is disconnected for every* $p \in \mathrm{M}$*,*

(3) M *is not locally connected at any point.*

Proof. Let $Y = l^\infty$ be the vector space of all bounded sequences of real numbers, with the supremum norm. Clearly Y is nonseparable and of size $\mathfrak{c}$. By Theorem 27, we obtain a function $\mathrm{M} \colon \mathbb{R} \to Y$ whose graph is connected and dense in $\mathbb{R} \times Y$. Let E be a connected subset of M containing two distinct points, say $a, b \in dom(E)$ with $a < b$. Then the set $E_0 = \mathrm{M} \cap ([a,b] \times Y)$ is contained in E because otherwise E would be disconnected, separated by $(-\infty, c) \times Y$ and $(c, \infty) \times Y$ for some $c \in (a,b)$. Since E_0 is dense in $[a,b] \times Y$, which is nonseparable, E is nonseparable too. That every point of M is a cut point — $\mathrm{M} \setminus \{p\}$ is disconnected for every $p \in \mathrm{M}$ — follows naturally from the fact that the domain of our function is the real line. Being discontinuous everywhere, M cannot be locally connected at any point because of Theorem 4. □

Thus we have constructed a connected metric space whose every nondegenerate connected subset is nonseparable. Other examples of such spaces are given in [10] and [11].

It still remains an open question whether there is a complete connected metric space that fails to be separably connected. We tried to prove that every complete connected metric space has to be separably connected but instead we came up with the following weaker result.

Theorem 29. *Let X be a reflexive Banach space. Suppose that $A \subset X$ is contained in the unit ball, connected in the norm topology (which implies that it is also connected in the weak topology) and closed in the weak topology (which implies that it is also closed in the norm topology). Then A is separably connected in the weak topology.*

Proof. Let $a, b \in A$ be distinct points. We are going to construct a set $T \subset A$ with $a, b \in T$ that is connected and separable in the weak topology.

Take any $n \in \mathbb{N}$. Since A is connected, there are finitely many points $a_0, a_1, \dots, a_{\alpha(n)} \in A$ such that $a_0 = a, a_{\alpha(n)} = b$ and $\|a_i - a_{i+1}\| < 1/n$. Let $T_n = \bigcup_{i=1}^{\alpha(n)} \overline{a_{i-1}\ a_i}$, where $\overline{u\ v}$ denotes the closed line segment between u and v. Naturally, every T_n is a compact connected set contained in the unit ball. These sets are also compact and connected in the weak topology. In particular, they are closed in the weak topology.

Let $\mathcal{H}$ be the collection of all subsets of the unit ball that are closed in the weak topology. Let $\mathcal{G}$ be the smallest topology on $\mathcal{H}$ containing sets of the form $\{K \in \mathcal{H} \colon K \cap U \neq \emptyset\}$ and $\{K \in \mathcal{H} \colon K \subset U\}$, where $U \subset X$ is open in the weak topology. The topological space $(\mathcal{H}, \mathcal{G})$ is the Vietoris topology for the unit ball with respect to the weak topology. Since X is reflexive, the unit ball is compact in the weak topology and therefore $(\mathcal{H}, \mathcal{G})$ is compact — see [7]. Hence we obtain a set $T \in \mathcal{H}$ which is a limit point — with respect to $\mathcal{G}$ — of the sequence of sets $T_n \in \mathcal{H}$. More precisely, $T \in \bigcap_{k\in\mathbb{N}} \overline{\{T_n \colon n \geq k\}} \subset \mathcal{H}$. Since the unit ball, being compact Hausdorff, is normal in the weak topology, the set T is connected in the weak topology — see [7].

We will argue by contradiction that $a, b \in T$. Suppose that $a \notin T$. Let $H = X \setminus \{a\}$. Then the set $\mathbf{H} = \{K \in \mathcal{H} \colon K \subset H\}$ is an open neighborhood of T in $\mathcal{H}$ but no set T_n belongs to this neighborhood because they all contain a. So $a \in T$ because otherwise T would not be a limit point of the sequence T_n. Similarly, $b \in T$.

In order to conclude that T is separable we will show that it is contained in the separable subspace $S = \overline{\mathrm{lin}(\bigcup_{n\in\mathbb{N}} T_n)}$. Suppose that there is a point $x \in T \setminus S$. Since S is a closed linear subspace with $x \notin S$, by the Hahn-Banach theorem, there is a continuous linear functional $\Lambda \in X^*$ such that $\Lambda(x) = 1$ and $\Lambda|_S = 0$. Let $U = \{z \in X \colon \Lambda(z) > 1/2\}$. Let $\mathbf{U} = \{K \in \mathcal{H} \colon K \cap U \neq \emptyset\}$. Since $x \in T \cap U$, $\mathbf{U}$ is an open neighborhood of T in $\mathcal{H}$. Since T is a limit point of the sequence T_n, there is an index $k \in \mathbb{N}$ such that $T_k \in \mathbf{U}$ and thus $T_k \cap U \neq \emptyset$, which yields a point $z \in S \cap U$, implying both $\Lambda(z) = 0$ and $\Lambda(z) > 1/2$. This contradiction shows that $T \subset S$ and thus T is separable. It follows that T is separable in the weak topology, too.

Let us argue by contradiction that $T \subset A$. Suppose that there is a point $x \in T \setminus A$. Now, since A is closed in the weak topology, there is a set U containing x and disjoint from A such that $U = \bigcap_{i=1}^{N} \{z \in X : |\Lambda_i(z) - \Lambda_i(x)| < \varepsilon\}$ where $\varepsilon > 0$ and $\Lambda_1, \dots, \Lambda_N \in X^*$ with $\|\Lambda_i\| = 1$. From the way the sets T_n were constructed it follows that there is a $k \in \mathbb{N}$ such that

$$(\forall n \geq k)(\forall t \in T_n)(\exists a \in A)\ \|t - a\| \leq \varepsilon/2.$$

Let $U' = \bigcap_{i=1}^{N} \{z \in X : |\Lambda_i(z) - \Lambda_i(x)| < \varepsilon/2\}$ and let $\mathbf{V} = \{K \in \mathcal{H} : K \cap U' \neq \emptyset\}$. Since $x \in T \cap U'$, $\mathbf{V}$ is a neighborhood of T and thus there is an index $n \geq k$ such that $T_n \in \mathbf{V}$, which yields a point $t \in T_n \cap U'$. Choose a point $a \in A$ with $\|t - a\| \leq \varepsilon/2$. Since $U \cap A = \emptyset$, $a \notin U$ and thus $|\Lambda_i(a) - \Lambda_i(x)| \geq \varepsilon$ for some i. Since $\|\Lambda_i\| = 1$, $|\Lambda_i(t) - \Lambda_i(a)| \leq \|t - a\| \leq \varepsilon/2$. Thus, $|\Lambda_i(t) - \Lambda_i(x)| \geq |\Lambda_i(a) - \Lambda_i(x)| - |\Lambda_i(t) - \Lambda_i(a)| \geq \varepsilon - \varepsilon/2 = \varepsilon/2$, and consequently $t \notin U'$. This contradiction shows that $T \subset A$. □

4.3 Products of Functions with Closed Connected Graphs

We are going to construct a complete separable connected metric space with singletons being its only connected compact subsets — as a product of countably many functions with closed connected graphs from the real line into a complete metric space. For this purpose, we need to develop some tools to ensure that an appropriately chosen sequence of functions with closed connected graphs again has a closed connected graph.

Let us first recall the notion of the product of an arbitrary family of functions defined on the same domain. Let X and $\{Y_i\}_{i \in I}$ be arbitrary sets. Let $f_i : X \to Y_i$ be a family of arbitrary functions indexed with $i \in I$. Let $Y = \Pi_{i \in I} Y_i$ be the Cartesian product of the family $\{Y_i\}_{i \in I}$. Then the product of the family of functions $\{f_i\}_{i \in I}$ will be a function $F : X \to Y$ denoted as $F = \bigotimes_{i \in I} f_i$ and defined by

$$F(x) = f_i(x)_{i \in I}$$

for every $x \in X$.

The following theorem establishes the basic facts concerning the product of a family of functions from a topological space into topological spaces. In particular, it serves to show that if all factor functions have closed graphs, then their product also has a closed graph.

Theorem 30. *Let X and $\{Y_i\}_{i \in I}$ be arbitrary topological spaces. Let $f_i : X \to Y_i$ be arbitrary functions. Endow $Y = \Pi_{i \in I} Y_i$ with the product topology. Let $F : X \to Y$ be given by $F = \bigotimes_{i \in I} f_i$. Then F is continuous at x_0 if and only if f_i is continuous at x_0 for each $i \in I$. Moreover, if f_i has a closed graph for each $i \in I$, then F has a closed graph.*

Proof. The proof is elementary so we omit it. □

The next two technical lemmas will be used to show that the product of an appropriately selected family of functions with connected graphs also has a connected graph.

Lemma 31. *Let $\{Y_i\}_{i\in I}$ be arbitrary topological spaces. Let $f_i\colon \mathbb{R} \to Y_i$ be functions with connected graphs such that f_i is discontinuous only at one point $a_i \in \mathbb{R}$ with $a_i = a_j \iff i = j$. Let $Y = \Pi_{i=1}^{\infty} Y_i$ be equipped with the product topology. Let $F\colon \mathbb{R} \to Y$ be given by $F = \bigotimes_{i\in I} f_i$. Then $(x, F(x)) \in \overline{F|_{(-\infty,x)}} \cap \overline{F|_{(x,+\infty)}}$ for every $x \in \mathbb{R}$.*

Proof. By Theorem 30, F is continuous on $\mathbb{R}\setminus\{a_i : i \in I\}$, so it suffices to write the proof for a point of discontinuity, say $x_0 = a_j$. Notice that f_i is continuous at x_0 whenever $i \neq j$. Let $g = \bigotimes_{i\in I\setminus\{j\}} f_i$. If we write $Y^{-j} = \Pi_{i\in I\setminus\{j\}} Y_i$, then, by Theorem 30, the function $g\colon X \to Y^{-j}$ is continuous at x_0.

Take any open set $U \subset Y$ with $F(x_0) \in U$ and any $r > 0$. We are done as soon as we find a point $a \in (x_0 - r, x_0)$ with $F(a) \in U$ and a point $b \in (x_0, x_0 + r)$ with $F(b) \in U$. We may assume — without loss of generality — that $U = \bigcap_{i\in I_0}\{h \in Y : h(i) \in U_i\}$, where U_i is open in Y_i for each $i \in I_0$ and I_0 is a finite subset of I. If $j \notin I_0$, we may put $U_j = Y_j$ and assume that $j \in I_0$. In any case, $f_j(x_0) \in U_j$.

Since $F(x_0) \in U$ and $g(x_0)$ is, in fact, a subset of $F(x_0)$, it follows that $g(x_0)$ belongs to $\bigcap_{i\in I_0\setminus\{j\}}\{h \in Y^{-j} : h(i) \in U_i\}$, which is an open subset of Y^{-j}. Since g is continuous at x_0, there is a $\delta \in (0, r)$ such that $f_i(x) \in U_i$ whenever $i \in I_0 \setminus \{j\}$ and $x \in (x_0 - \delta, x_0 + \delta)$.

Since f_j has a connected graph, there is a point $a \in (x_0 - \delta, x_0)$ with $f_j(a) \in U_j$ and a point $b \in (x_0, x_0 + \delta)$ with $f_j(b) \in U_j$. Thus $F(a) \in U$ and $F(b) \in U$, which completes the proof. □

The following lemma is about a function from the real line whose set of all discontinuity points has smaller cardinality than the real line.

Lemma 32. *Let Y be an arbitrary topological space. Let $F\colon \mathbb{R} \to Y$ be continuous on E with $|\mathbb{R} \setminus E| < \mathfrak{c}$. Assume that $(x, F(x)) \in \overline{F|_{(-\infty,x)}} \cap \overline{F|_{(x,+\infty)}}$ for every $x \in \mathbb{R}$. Then F has a connected graph.*

Proof. Suppose that F is disconnected. Then it is the union of two disjoint nonempty relatively closed sets F_1 and F_2. Let $A = dom(F_1)$ and $B = dom(F_2)$. Then $\mathbb{R} = A \cup B$ with $A \cap B = \emptyset$ and the sets A, B are nonempty. Since $\mathbb{R}$ is connected, $\overline{A}\cap\overline{B} \neq \emptyset$. We will show that $\overline{A}\cap\overline{B}$ is contained in $\mathbb{R}\setminus E$ to conclude that — being a closed set — it has to be countable. Take any $x \in \overline{A} \cap \overline{B}$. Then there are two sequences $a_n \in A$ and $b_n \in B$ converging to x. If x were a point of continuity, it would follow that the point $(x, F(x))$ — being the limit of both $(a_n, F(a_n)) \in F_1$ and $(b_n, F(b_n)) \in F_2$ — belongs to $F_1\cap F_2$, which is impossible. So the closed set $\overline{A}\cap\overline{B}$ is countable and thus has an isolated point, say $x \in \overline{A}\cap\overline{B}$ with $(x-r, x)\cup(x, x+r) \subset \mathbb{R}\setminus(\overline{A}\cap\overline{B}) = Int(\mathbb{R}\setminus A)\cup Int(\mathbb{R}\setminus B) = Int(B)\cup Int(A)$. Now, the connected set $(x, x+r)$ — covered by the union of two disjoint open sets $Int(A) \cup Int(B)$ — must be contained in one of them. The same argument goes for $(x-r, x)$. Let us assume — without loss of generality — that $x \in A \cap \overline{B}$. Then it follows that $(x-r, x) \subset B$ or $(x, x+r) \subset B$. In either case, $(x, F(x)) \in \overline{F|_B}$ and, as before, it follows that $(x, F(x)) \in F_1 \cap F_2$. This contradiction shows that F has a connected graph. $\square$

4.4 Completely Metrizable Connected Punctiform Space

At the heart of our construction of a complete separable connected punctiform space is a function from the real line into a complete metric space with a closed connected graph and discontinuous only at one point. That such a function exists is indeed lucky for our construction, because the condition of having a closed connected graph is close to entailing continuity in certain natural contexts — see Corollary 19 in the previous chapter.

Theorem 33. *There exists a complete separable connected metric space* D *with the following properties:*

(1) each connected compact subset of D *is a singleton,*

(2) $\mathrm{D} \setminus \{p\}$ *is disconnected for every* $p \in \mathrm{D}$*,*

(3) D *is locally connected at each point of a dense set* E *with* $D \setminus E$ *countable.*

Proof. The basic building block of this construction is a function from the real line into some complete metric space with a closed connected graph discontinuous at one point only. Any such function will do. For instance, let $f\colon \mathbb{R} \to Y$ be as in Example 20, discontinuous only at 0. Notice that the graph of f is separable even if Y is not. Let $\mathbb{Q} = \{a_1, a_2, \ldots\}$ with $a_i = a_j$ if and only if $i = j$. Let $f_i\colon \mathbb{R} \to Y$ be given by $f_i(x) = f(x - a_i)$ for every $i \in \mathbb{N}$. Let $Y' = Y^{\mathbb{N}}$ with a complete product metric. Let $F\colon \mathbb{R} \to Y'$ be given by $F(x) = (f_1(x), f_2(x), \ldots)$. By Theorem 30, F has a closed graph and is continuous on $E = \mathbb{R} \setminus \mathbb{Q}$. By Lemma 31 and Lemma 32, F has a connected graph. We claim that the graph of F with the induced metric from $\mathbb{R} \times Y'$ is the desired space. Clearly, F is a complete connected metric space satisfying (2). To see that F is separable let us embed it in the separable metric space $Z = \Pi_{i=1}^{\infty} f_i$ in the following way $(x, F(x)) \longleftrightarrow \big((x, f_1(x)), (x, f_2(x)), \ldots\big)$.

Let E be a connected subset of F containing two distinct points $a, b \in dom(E)$ with $a < b$. Then the set $E_0 = F \cap ([a, b] \times Y')$ is contained in E because otherwise E would be disconnected, separated by $(-\infty, c) \times Y'$ and $(c, \infty) \times Y'$ for some $c \in (a, b)$. Choose a rational number $q \in (a, b)$ — a point of discontinuity of function F. Then, since the graph of F is closed, there is a sequence $(x_n, F(x_n)) \in E_0$ such that $x_n \to q$ and $F(x_n)$ has no subsequential limit. So E is not compact.

To prove (3) we make use of our characterization of continuity for functions from the real line with connected graphs — Theorem 4 — which in our case comes down to the observation that F is locally connected at $(x, F(x))$ if and only if F is continuous at x. $\square$

The space D constructed above is a closed subset of $(l^2)^{\mathbb{N}}$. According to [1] the following three spaces are homeomorphic: $(l^2)^{\mathbb{N}}$, l^2, $\mathbb{R}^{\mathbb{N}}$. Therefore our completely metrizable connected punctiform space can in fact be thought of as a closed subset of l^2 or as a closed subset of $\mathbb{R}^{\mathbb{N}}$. There is an old classical example of a function $f\colon \mathbb{R} \to \mathbb{R}$ with a G_δ connected punctiform graph, [6].

4.5 Extremal Functions

The concept of a separably connected space arose in the course of investigating the question whether a continuous real-valued function — defined on a connected space — having a local extremum everywhere (without knowing for a given point whether it is a maximum or a minimum) has to be constant.

A positive answer is given for the class of separably connected metric spaces. However, it is rather natural for a nonmetrizable connected space to entirely fail to be separably connected and to admit a nonconstant continuous function with a local extremum everywhere, so the question whether such functions must be constant has to be limited to metric spaces.

At this point it might be useful to note briefly that if X is connected and $f\colon X \to \mathbb{R}$ is lower semicontinuous with a local maximum everywhere then it must be constant because the set $\{x \in X\colon f(x) \leq f(a)\}$ is clopen and nonempty for any $a \in X$. Therefore, in this context, it is intriguing to study only such functions which have a local extremum at every point without knowing whether it is a maximum or a minimum, which will be called *extremal functions*.

Naturally, in order to expect an extremal function to be constant one has to impose some kind of connectedness condition on its graph, but not necessarily continuity. In fact, we have shown that every extremal Darboux function defined on a separably connected metric space has to be constant.

It is still an open question whether a Darboux (or continuous) extremal function defined on a connected, but not separably connected, metric space must be constant. Any theorem in this field must not include among its assumptions both completeness and local connectedness because of the classical theorem stating that every complete connected and locally connected metric space is arcwise connected, and thus separably connected — see [3] section 6.3.11.

A space is *Baire* if all of its nonempty open subsets are of second category. Let us call a space *strongly Baire* if it is Baire and all of its closed subsets are Baire. We have proved that extremal continuous functions are constant on connected and locally connected strongly Baire metric spaces. Although every topologically complete space is strongly Baire, there exists a certain separable connected and locally connected strongly Baire metric space which is not topologically complete.

Let us begin with a simple example which immediately brings the context into metric spaces.

Example 34. *There is a "very nice" nonmetrizable compact connected linearly ordered topological space X, which is not separably connected, and a continuous nonconstant function $f\colon X \to \mathbb{R}$ with a local extremum everywhere.*

Proof. Endow $X = [0,1] \times [2,3]$ with the linear lexicographical order and with the order topology. The end points of X, namely $(0,2)$ and $(1,3)$, cannot be contained in a separable connected subset of X, because such a connected set would have to be the whole X, which is nonseparable, as the open intervals $](x,2),(x,3)[$ indexed with $x \in [0,1]$ form an uncountable family of disjoint open nonempty sets. Let $f\colon X \to \mathbb{R}$ be given by $f(x,y) = x$. Then f is a continuous nonconstant function with a local extremum at every point. $\square$

The following theorem, combined with the fact that separable metric spaces are second countable, constitutes the core of the argument that on a separably connected metric space a Darboux function with a local extremum everywhere has to be constant.

Theorem 35. *Let X be a second countable space. Assume that $f\colon X \to \mathbb{R}$ has a local extremum everywhere. Then $f(X)$ is countable.*

Proof. For any open set $U \subset X$ let us write

$$max(U) = \{x \in U\colon f(z) \leq f(x) \text{ for all } z \in U\},$$

$$min(U) = \{x \in U\colon f(z) \geq f(x) \text{ for all } z \in U\}.$$

Notice that $f(max(U))$ and $f(min(U))$ are singletons or empty. Let $U_1, U_2, \ldots$ be a countable basis. Since f has a local extremum everywhere,

$$X = \bigcup_{n\in\mathbb{N}} max(U_n) \cup min(U_n).$$

Hence $f(X) = \bigcup_{n\in\mathbb{N}} f(max(U_n)) \cup f(min(U_n))$ is a countable union of singletons. $\square$

Keeping the previous theorem in mind, we argue that Darboux extremal functions are constant provided they are defined on spaces such that any two points can be contained in a connected second countable subset. In the case of metric spaces, we are talking about separable connectedness.

Theorem 36. *Let X be a separably connected metric space. Let $f\colon X \to \mathbb{R}$ be a Darboux function with a local extremum everywhere. Then f is constant.*

Proof. Take any two points $a, b \in X$. Since X is separably connected, there is a separable connected set $K \subset X$ with $a, b \in K$. Since K is metrizable, it is second countable, and by Theorem 35, $f(K)$ is countable. Since f is Darboux, $f(K)$ is connected. As a connected countable metrizable set, $f(K)$ is a singleton, so $f(a) = f(b)$. Thus f is constant. $\square$

What follows is an attempt at producing an alternative argument — without recourse to separability — that continuous extremal functions defined on connected metric spaces ought to be constant.

Theorem 37. *Let (X,d) be a Baire metric space. Let $f\colon X \to \mathbb{R}$ be a continuous function with a local extremum everywhere. Then f is constant on some ball. Moreover, f is locally constant on a dense open subset.*

Proof. Let

$$\Delta_r^{max} = \{x \in X\colon (\forall z \in B(x,r))\ f(z) \le f(x)\},$$
$$\Delta_r^{min} = \{x \in X\colon (\forall z \in B(x,r))\ f(z) \ge f(x)\}.$$

Notice that

$$\Delta_r^{max} = \bigcap_{z\in X} \{x \in X\colon d(x,z) \ge r\} \cup \{x \in X\colon f(z) \le f(x)\}.$$

Hence, since f is continuous, the sets Δ_r^{max} and Δ_r^{min} are closed. Since f has a local extremum everywhere, we can write the Baire space X as a countable union of closed sets in the following way:

$$X = \bigcup_{n\in\mathbb{N}} \Delta_{1/n}^{max} \cup \bigcup_{m\in\mathbb{N}} \Delta_{1/m}^{min}.$$

So, there exist $N \in \mathbb{N}$, $x_0 \in X$, $r > 0$ such that, say, $B(x_0,r) \subset \Delta_{1/N}^{max}$. Assuming that $2r < 1/N$ it is easy to see that $f(B(x_0,r)) \subset \{f(x_0)\}$.

Naturally, since X is Baire, we can repeat this argument for any open ball, showing that f is locally constant on a dense open set. □

In the following theorem, we assume that the domain is strongly Baire, which is essentially weaker than topological completeness, even among connected and locally connected metric spaces.

Theorem 38. *Let X be a connected and locally connected metric space, whose each closed subspace is Baire. Let $f\colon X \to \mathbb{R}$ be a continuous function with a local extremum everywhere. Then f is constant.*

Proof. Let $B = \bigcup\{J \subset X\colon J \text{ is open and } f|_J \text{ is constant}\}$. By Theorem 37, B is dense. Let us argue by contradiction that $B = X$. Suppose that this is not the case. Then the closed nonempty subset $A = X \setminus B$ is a Baire space. Hence by Theorem 37, there is an open set $U \subset X$ and a point $a \in A \cap U$ such that $f(U \cap A) = \{f(a)\}$. Since X is locally connected, U can be assumed to be connected. Since $a \notin B$, there is a point $q \in U$ such that $f(q) \neq f(a)$. Since f is continuous and since B is dense, there is a point $b \in B \cap U$ such that $f(b) \neq f(a)$. Let $B_0 = \bigcup\{J\colon J \text{ is open and } f(J) = \{f(b)\}\}$. Now we have the following two claims concerning the set B_0:

(1) $(\overline{B_0} \setminus B_0) \subset A$,

(2) $(\overline{B_0} \setminus B_0) \cap U \neq \emptyset$.

To show that (1) holds assume that there is a point $z \in ((\overline{B_0} \setminus B_0)) \setminus A$. Then $z \in B$ and we have an open set V such that $z \in V$ and $f(V) = \{f(z)\}$. Furthermore, since $z \in \overline{B_0}$, $f(z) = f(b)$, and thus $V \subset B_0$ and consequently $z \in B_0$. But $z \notin B_0$.

To show that (2) holds assume that $(\overline{B_0} \setminus B_0) \cap U = \emptyset$. Then the set $U \cap B_0 = U \cap \overline{B_0}$ is clopen in U. It contains b but it doesn't contain a. Since $a, b \in U$ and U is connected, this yields a contradiction.

From (1) $\wedge$ (2) it follows that there is a point $z \in (A \cap U) \cap \overline{B_0}$. Hence $f(z) = f(a)$ and $f(z) = f(b)$, but $f(a) \neq f(b)$. This contradiction finally shows that $X = B$.

Fix a point $b \in B$. Let $B' = \mathrm{Int}(\{x \in X \colon f(x) = f(b)\})$. Naturally, $b \in B'$ and B' is open. We will show that it is also closed. Take any $x \in \overline{B'}$. Since $x \in B$, there is an open set V containing x such that $f(V) = \{f(x)\}$. Since $x \in \overline{B'}$, there is a point $z \in V \cap B'$. Hence $f(z) = f(x) = f(b)$. In effect, $f(V) = \{f(b)\}$ and thus $x \in B'$. We showed that $\overline{B'} \subset B'$. So B' is a nonempty clopen subset of the connected space X. Hence $X = B'$ and f is constant. $\square$

Let us show that there exists a connected, locally connected strongly Baire metric space that is not topologically complete. We are going to use the fact that every separable topologically complete space with no isolated points contains a compact set of size $\mathfrak{c}$. We will find it convenient to say that a subset of a topological space is *sprawled* if it intersects every compact set of size $\mathfrak{c}$. Moreover, if both A and $X \setminus A$ are sprawled in X, then we will say that A is a Bernstein subset of X — in conformance with existing terminology. Naturally, a Bernstein subset of a separable complete metric space with no isolated points is not topologically complete. We will show that any subset that is sprawled in a separable complete metric space with no isolated points is strongly Baire. It only remains to realize that a Bernstein subset of the Euclidean plane is connected and locally connected.

Let X be a separable complete metric space with no isolated points and let K be sprawled in X. To show that K is strongly Baire we consider an arbitrary relatively closed subset K_0 of K and argue that K_0 is Baire. Let $K_2 = \{x \in K_0 \colon x \text{ is an isolated point of } K_0\}$. Let $K_1 = K_0 \setminus K_2$. K_2 is open in K_0, hence K_1 is closed in K_0. Thus K_1 is closed in K and has no isolated points. Let us write $K_1 = K \cap F$, where $F = \overline{K_1} = Clo_X(K_1)$. So F is closed in X and has no isolated points. Thus F is separable complete with no isolated points. Let G_0 be an arbitrary nonempty relatively open subset of K_1. Then $G_0 = K_1 \cap G$, for some open subset G of X. Now, the set $F \cap G$ is open in F, hence $F \cap G$ is topologically complete with no isolated points. Let $A_n \subset K_1$ be an arbitrary sequence of sets which are nowhere dense in K_1. Then they are nowhere dense in F, because $K_1 \subset F$. Actually, they are also nowhere dense in $F \cap G$. Therefore the set $A = \bigcup_n Clo_{F \cap G}(A_n)$ has empty interior in $F \cap G$, because $F \cap G$ is Baire. It follows that $(F \cap G) \setminus A$ is a dense G_δ subset of $F \cap G$ with no isolated points. Now, since $(F \cap G) \setminus A$ is topologically complete with no isolated points, it contains a compact set of size $\mathfrak{c}$, say $P \subset (F \cap G) \setminus A$. Since K is sprawled in X, there is a point $z \in K \cap P \subset (K \cap F \cap G) \setminus A = G_0 \setminus A$, which means that G_0 is not of first category in K_1. Thus we showed that K_1 is Baire. Since K_2 is a discrete space, $K_0 = K_1 \cup K_2$ is Baire, completing the proof that K is strongly Baire.

Appendix A

Jelinek's Discontinuous Function with a Closed Connected Graph

A.1 Construction of The Graph

Demarcating the Domain

We are going to need two functions a, r with the following properties:

1. $0 \swarrow a(n)$ as $n \to \infty$
2. $a(k_1, k_2, ..., k_N) \swarrow a(k_1, k_2, ..., k_N, n)$ as $n \to \infty$
3. $0 < r(k_1, \ldots, k_N) < min \left\{ \frac{1}{k_1}, \ldots, \frac{1}{k_N}, \frac{1}{2^N} \right\}$
4. $0 < a(n+1) - r(n+1) < a(n+1) + r(n+1) <$
   $< a(n) - r(n) < a(n) + r(n) < 1/2$
5. $a(k_1, \ldots, k_N) <$
   $< a(k_1, \ldots, k_N, n+1) - r(k_1, \ldots, k_N, n+1) <$
   $< a(k_1, \ldots, k_N, n+1) + r(k_1, \ldots, k_N, n+1) <$
   $< a(k_1, \ldots, k_N, n) - r(k_1, \ldots, k_N, n) <$
   $< a(k_1, \ldots, k_N, n) + r(k_1, \ldots, k_N, n) <$
   $< a(k_1, \ldots, k_N) + r(k_1, \ldots, k_N)/2.$

for every $n, N \in \mathbb{N}$ and every $k_1, k_2, ..., k_N \in \mathbb{N}$.

For $a = a(k_1, k_2, ..., k_N)$ and $r = r(k_1, k_2, ..., k_N)$ define

$$U(k_1, k_2, ..., k_N) =$$

$((a-r,r)\times(r,2^{-N}+r))\cup(\{a\}\times(2^{-N},2^{-N}+r))\cup((a,a+r)\times(0,2^{-N}+r))$.

Notice that

$z\in U(k_1,\ldots,k_N)\Rightarrow dist(z,\delta(U(k_1,\ldots,k_N))\le r(k_1,\ldots,k_N)$.

$$\text{Let } W=(0,1)^2\setminus\bigcup_{n=1}^{\infty}U(n).$$

$$\text{Let } W(k_1,...,k_N)=U(k_1,...,k_N)\setminus\bigcup_{n=1}^{\infty}U(k_1,...,k_N,n).$$

Let $A^*=\{\ (a(k_1,...,k_N),\ y)\ :\ N,k_1,...,k_N\in\mathbb{N}\wedge 0<y\le 2^{-N}\ \}$.
Let $K=\delta([0,1]^2)$. Let $A=\{(k_1,...,k_N)\colon N,k_1,k_2,...,k_N\in\mathbb{N}\}$. Then

$$W\cup\bigcup_{a\in A}W(a)=(0,1)^2.$$

Definition of the Function

We are going to define a function $f\colon[0,1]^2\to[0,\infty)$ in the following way. Let $f(1,y)=f(x,0)=f(x,1)=1$ for $x,y\in[0,1]$ and let

$$f(0,y)=1/y\quad\text{for } y\in(0,1].$$

For $(a,y)\in A^*$ let

$$f(a,y)=1/y.$$

For $(x,y)\in W\setminus A^*$ let

$$f(x,y)=\frac{1}{dist(\ (x,y),\ \delta([0,1]^2)\ )}.$$

Notice that

$$((0,1)^2\setminus W)\setminus A^*=\bigcup_{a\in A}W(a)\setminus A^*,$$

where $\{W(a)\colon a\in A\}$ is a family of pairwise disjoint sets.

For $(x,y)\in W(k_1,...,k_N)$ let

$$f(x,y)=\frac{1}{dist(\ (x,y),\ \delta(U(k_1,...,k_N))\ )}.$$

This definition is correct because the sets $U(\ldots)$ are open.

We will show that this function $f\colon[0,1]^2\to[0,\infty)$ has a closed connected graph. It can be extended to the whole plane by putting

$$f(x,y)=\begin{cases}f(-x,y), & (x,y)\in[-1,0]\times[0,1],\\ 1 & (x,y)\notin[-1,0]\times[0,1].\end{cases}$$

A.2 The graph is closed

We will split the domain of the function into several kinds of sets and for each kind we will provide a separate proof that the graph is closed:

$$[0,1]^2 = K \cup W \cup \bigcup_{a\in A} W_a =$$

$$= K \cup A^* \cup Int(W) \cup \bigcup_{n=1}^{\infty} \delta(U(n)) \cup \bigcup_{a\in A} [Int(W_a) \cup \bigcup_{n=1}^{\infty} \delta(U(k(a),n)]$$

$$= K \cup A^* \cup Int(W) \cup \bigcup_{a\in A} Int(W_a) \cup \bigcup_{a\in A} \delta(U_a).$$

If $(x_n, y_n) \to (x, 0)$ with $y_n > 0$ then $f(x_n, y_n) = y_n^{-1}$ or $f(x_n, y_n) = dist(\ldots)^{-1}$, in which case $dist(\ldots) \le y_n$. In any case $f(x_n, y_n) \ge y_n^{-1}$ and so $f(x_n, y_n) \to \infty$. Hence the graph of f is closed at every point $(x, 0)$. It is quite easy to see that the graph is closed at each point in $[0,1] \times \{1\} \cup \{1\} \times [0,1]$.

Let (x, y) be an arbitrary point in A^*: $x = a(k_1, \ldots, k_N)$ and $0 < y \le 2^{-N}$. We will examine all possible sequences (x_n, y_n) converging to (x, y):

1. if $(x_n, y_n) \in A^*$ then $f(x_n, y_n) = {y_n}^{-1} \to y^{-1} = f(x, y)$,
2. if $(x_n, y_n) \in W(k_1, \ldots, k_N) \setminus A^*$ then
$f(x_n, y_n) = dist(\ (x, y),\ \delta(U(k_1, ..., k_N))\ )^{-1} = |x_n - x|^{-1} \to \infty$,
3. if $(x_n, y_n) \in W(k_1, \ldots, k_N, k_{N+1}(n), \ldots)$ then
$f(x_n, y_n) = dist(\ (x_n, y_n),\ \delta(U(k_1, ..., k_N, k_{N+1}(n), \ldots))\ )^{-1}$
$\ge r(k_1, \ldots, k_N, k_{N+1}(n), \ldots)^{-1} \ge k_{N+1}(n) \to \infty$,
4. if $(x_n, y_n) \in W$ or $(x_n, y_n) \in W(k_1, \ldots, k_{N-1})$ then $f(x_n, y_n) \to f(x, y)$.

Hence the graph of f is closed at every point in A^*. Analogously, we convince ourselves that the graph is closed at every point $(0, y)$. Thus the graph is closed at every point in $K \cup A^*$. Furthermore, if $z \in Int(W) \cup \bigcup_{a\in A} Int(W_a)$ then f is clearly continuous at z, hence closed.

Let $z \in \delta U_a \setminus A^*$ for some $a \in A$. We will examine all possible sequences z_n converging to z. If $z_n \in U_a$ then $f(z_n) = dist(z_n, \delta U_a)^{-1} \to \infty$, and if $z_n \notin U_a$ then $f(z_n) \to f(z)$. Hence the graph is closed at z. Thus we have completed the proof that the graph of f is closed.

A.3 The graph is connected

The proof that the graph of f is connected is based on the following scheme

1. $[0,1]^2 = K \cup W \cup \bigcup_{a\in A} W_a$
2. $Gr(f|_K) \cup Gr(f|_W)$ is connected
3. $Gr(f|_W) \cup Gr(f|_{W(n)})$ is connected
4. $Gr(f|_{W(k_1,\ldots,k_N)}) \cup Gr(f|_{W(k_1,\ldots,k_N,n)})$ is connected

for every $N \in \mathbb{N}$, every $k_1, \ldots, k_N \in \mathbb{N}$, and every $n \in \mathbb{N}$.

Evidently, $Gr(f|_K)$ is connected. We will show that the sets $Gr(f|_W)$, $Gr(f|_{W(k_1,\ldots,k_N)})$ are connected by pointing out that the sets W, $W(k_1,\ldots,k_N)$ are connected and proving that f restricted to any of these sets is continuous. Points (2)-(4) will be proved by using the following lemma.

Lemma 39. *If A, B are connected and $A \cap \overline{B} \neq \emptyset$ then $A \cup B$ is connected.*

Notice that

$$Gr(f|_K) \ni (0, 1/2, 2) = lim_{n\to\infty}(a(n), 1/2, 2),$$

$$(a(n), 1/2, 2) \in Gr(f|_W).$$

Hence

$$(0, 1/2, 2) \in Gr(f|_K) \cap \overline{Gr(f|_W)}.$$

So $Gr(f|_K) \cup Gr(f|_W)$ is connected.
Similarly, we show that $Gr(f|_W) \cup Gr(f|_{W(k_1)})$ is connected for each $k_1 \in \mathbb{N}$:

$$Gr(f|_W) \ni (a(k_1), 1/4, 4) = lim_{n\to\infty}(a(k_1, n), 1/4, 4),$$

$$(a(k_1, n), 1/4, 4) \in Gr(f|_{W(k_1)}).$$

Hence

$$(a(k_1), 1/4, 4) \in Gr(f|_W) \cap \overline{Gr(f|_{W(k_1)})}.$$

So $Gr(f|_W) \cup Gr(f|_{W(k_1)})$ is connected. Thus points (2) and (3) have been demonstrated and (4) can be done analogously. To complete the proof that the graph of f is connected we still need to prove the continuity of $f|_W$ and $f|_{W(k_1,\ldots,k_N)}$. We will skip $f|_W$ and focus on $f|_{W(k_1,\ldots,k_N)}$ because $f|_W$ can be done analogously.

We are going to show that f restricted to $W(k_1,\ldots,k_N)$ is continuous by using the following lemma.

Lemma 40. *If E, H are closed, $g\colon E \cup H \to Y$ and $g|_E$, $g|_H$ are continuous then g is continuous.*

Let us define the two sets H and E as follows:

$$H = \bigcup_{n=1}^{\infty} \{a(k_1, \dots, k_N, n)\} \times [a(k_1, \dots, k_N, n) - a(k_1, \dots, k_N), 2^{-N}],$$

$$E = W(k_1, \dots, k_N) \setminus \bigcup_{n=1}^{\infty} H_n, \text{ where}$$

$$H_n = \{a(k_1, \dots, k_N, n)\} \times (a(k_1, \dots, k_N, n) - a(k_1, \dots, k_N), 2^{-N}].$$

Notice that H is a relatively closed subset of $W(k_1, \dots, k_N)$. We will show that each H_n is a relatively open subset of $W(k_1, \dots, k_N)$ in order to conclude that E is a relatively closed subset of $W(k_1, \dots, k_N)$. Obviously, $W(k_1, \dots, k_N) = E \cup H$. Notice that $f|_H(x, y) = y^{-1}$, hence $f|_H$ is continuous. We will show that $f|_E(x, y) = dist(\ (x, y),\ \delta(U(k_1, ..., k_N))\ )^{-1}$ to conclude that $f|_E$ is continuous. All of this put together implies that f restricted to $W(k_1, \dots, k_N)$ is continuous. The reasoning above is made complete by the following claims and their proofs.

Claim 41. *The set*

$$H_n = \{a(k_1, \dots, k_N, n)\} \times (a(k_1, \dots, k_N, n) - a(k_1, \dots, k_N), 2^{-N}]$$

is a relatively open subset of $W(k_1, \dots, k_N)$.

Proof. Let us simplify notation as follows. Let $a' = a(k_1, \dots, k_N)$.
Let $a = a(k_1, \dots, k_N, n)$. Let $r = r(k_1, \dots, k_N, n)$.
Let $W_a = W(k_1, \dots, k_N)$. Let $U_a = U(k_1, \dots, k_N, n)$.
Let (a, y_0) be an arbitrary element of H_n. Choose y' so that $a - a' < y' < y_0$. Let $G = (a - r, a + r) \times (y', y_0 + r)$. Our proof will be finished as soon as we show that $G \cap W_a \subset H_n$. Take any $(x, y) \in G \cap W_a$. Now, $a - r < x < a + r$. Since $(x, y) \in W_a$ it follows that $(x, y) \notin U_a$. Notice that

$$r < a - a' < y' < y < y_0 + r \leq 2^{-N} + r.$$

Now, if $a - r < x < a$ then $(x, y) \in (a - r, a) \times (r, 2^{-N} + r) \subset U_a$. On the other hand, if $a < x < a + r$ then $(x, y) \in (a, a + r) \times (0, 2^{-N} + r) \subset U_a$. Thus $x = a$. If $y > 2^{-N}$ then $(x, y) \in \{a\} \times (2^{-N}, 2^{-N} + r) \subset U_a$. Thus $y \leq 2^{-N}$. So $(x, y) \in H_n$. ☐

The following proposition will be used to show that
$f|_E(x, y) = dist(\ (x, y),\ \delta(U(k_1, ..., k_N))\ )^{-1}$.

Claim 42. *If* $0 < y < a(k_1, \dots, k_N, n) - a(k_1, \dots, k_N)$ *then*

$$y = dist(\ (a(k_1, \dots, k_N, n), y),\ \delta(U(k_1, \dots, k_N))\ ).$$

Proof. Let us simplify notation as follows.
Let $a' = a(k_1, \dots, k_N)$. Let $r' = r(k_1, \dots, k_N)$.
Let $a = a(k_1, \dots, k_N, n)$. Let $r = r(k_1, \dots, k_N, n)$.

Let $U_{a'} = U(k_1, \ldots, k_N)$.
Recall that $a - a' < r' < 2^{-N}$. Hence

$$0 < y < \frac{r' + 2^{-N}}{2} \quad \text{and} \quad 0 < y < 2^{-N}.$$

Recall that $a' < a < a' + r'/2$. Considering the shape of the boundary of $U_{a'}$, these three inequalities allow us to conclude that

$$dist((a, y), \delta(U_{a'})) = min\{a - a', y\} = y.$$

$\square$

Claim 43. $f|_E(x, y) = \dfrac{1}{dist(\ (x, y),\ \delta(U(k_1, \ldots, k_N))\ )}$.

Proof. Recall that $E \subset W(k_1, \ldots, k_N))$. Hence, for $(x, y) \in E \setminus A^*$ — by the definition of f — we have $f|_E(x, y) = dist(\ (x, y),\ \delta(U(k_1, \ldots, k_N))\ )^{-1}$. Notice that

$$E \cap A^* = \bigcup_{n=1}^{\infty} \{a(k_1, \ldots, k_N, n)\} \times (0, a(k_1, \ldots, k_N, n) - a(k_1, \ldots, k_N)].$$

We finish this proof by referring to the previous claim.

$\square$

A.4 The graph is not arcwise connected

Let $L = f|_{\{0\} \times (0,1/2]}$. Notice that $f|_K \setminus L$ is open in $Gr(f)$. Hence

$$(1)\ f|_K \cap \overline{f \setminus f|_K} \subset L.$$

Let us choose some $z_0 \in f|_K$ and $z_1 \in f \setminus f|_K$. If the graph of f were arcwise connected then there would exist a continuous function $h\colon [0, 1] \to Gr(f)$ such that $h(0) = z_0$ and $h(1) = z_1$. We will suppose that such a function exists and obtain a contradiction, thus proving that the graph is not arcwise connected. Let $E = h([0, 1])$. The set E has the following properties:

$$(2)\ E \text{ is bounded and connected,}$$

$$(3)\ E \cap f|_K \neq \emptyset \text{ and } E \setminus f|_K \neq \emptyset.$$

Moreover, we will show that

$$(4)\ E \cap f|_K \cap \overline{E \setminus f|_K} \neq \emptyset.$$

Let us argue by contradiction. If $E \cap f|_K \cap \overline{E \setminus f|_K} = \emptyset$, then $E \cap f|_K \subset E \setminus \overline{E \setminus f|_K} \subset E \cap f|_K$. Hence $E \cap f|_K = E \setminus \overline{E \setminus f|_K}$ is a relatively closed and relatively open subset of the connected set E. By (3), it is not empty and it is a proper subset of E. To obtain this contradiction we have used the fact that $f|_K$ is a relatively closed subset of $Gr(f)$. Now, putting (1) and (4) together we obtain a point z such that

$$(5)\ z \in E \cap f|_K \cap \overline{E \setminus f|_K} \cap L.$$

Independently of the reasoning above, there exists a $\delta > 0$ such that

$$(6)\ E \cap f|_{(0,\delta)\times(0,1)} \subset f|_{A^*}$$

because otherwise the set E would not be bounded. From (5) and (6) it follows that there exists a point $(a, y) \in A^* \cap (0, \delta) \times (0, 1)$ with $((a, y), f(a, y)) \in E$. Furthermore, there exists an $r > 0$ such that $a + r < \delta$ and

$$(7)\ A^* \cap [a, a + r] \times (0, 1) = A^* \cap (a - r, a + r) \times (0, 1).$$

By (6) and (7),

$$E \cap f|_{[a,a+r]\times(0,1/2]} = E \cap f|_{(a-r,a+r)\times(0,1)}.$$

Notice that the set above is nonempty because it contains $((a, y), f(a, y))$. Moreover, it is a relatively closed and relatively open proper subset of the connected set E. This contradiction completes the proof.

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