Metric spaces are paracompact

1. (X, d) is a metric space
2. $X = \bigcup_{s \in S} U_s$
3. U_s is open for each $s \in S$
4. S is well-ordered

By induction over $n \in \mathbb{N}$, define

$$V_{s,n} = \bigcup \{B(c, 1/2^n) \colon \Phi(s, n, c)\}$$

for all $s \in S$, $n \in \mathbb{N}$, where $\Phi(s, n, c) \iff (1) \wedge (2) \wedge (3)$:

(1) $c \in U_s \wedge (\forall t < s)(c \notin U_t)$

(2) $(\forall k < n)(\forall t \in S)(c \notin V_{t,k})$

(3) $B(c, 3/2^n) \subset U_s$.

The sets $V_{s,n}$ are open and $V_{s,n} \subset U_s$. For any $x \in X$, taking the smallest $s \in S$ with $x \in U_s$ and some $n \in \mathbb{N}$ with $B(x, 3/2^n) \subset U_s$, we have that

$$(\exists k < n)(\exists t \in S)(x \in V_{t,k}) \ \vee \ x \in V_{s,n}$$

showing that $X \subset \bigcup \{V_{s,n} \colon (s, n) \in S \times \mathbb{N}\}$.

For a fixed $n \in \mathbb{N}$, take any $x \in V_{s,n}$ and $y \in V_{t,n}$ with $s < t$. We get $a \in X$ with $x \in B(a, 1/2^n) \subset B(a, 3/2^n) \subset U_s$ and $b \in X$ with $y \in B(b, 1/2^n)$, $b \in U_t \setminus U_s$. Thus $d(a, b) \geq 3/2^n$ and $d(x, y) \geq d(a, b) - d(a, x) - d(b, y) > 1/2^n$. We have shown that

$$(\forall n \in \mathbb{N}) \ \ (x \in V_{s,n} \wedge y \in V_{t,n} \wedge s \neq t) \implies d(x, y) > 1/2^n,$$

so $\forall n \in \mathbb{N}$ the family $\mathcal{V}_n = \{V_{s,n} \colon s \in S\}$ is discrete.

Take any $x \in X$. We get k, j and t with $B(x, 1/2^k) \subset V_{t,j}$. Now, for any $n \geq j + k$, $s \in S$, $c \in X$ with $\Phi(s, n, c)$ we have that $c \notin V_{t,j}$ and thus $d(x, c) \geq 1/2^k$ ensuring that $B(x, 1/2^{j+k}) \cap B(c, 1/2^n) = \emptyset$. Consequently, $B(x, 1/2^{j+k}) \cap V_{s,n} = \emptyset$ for all $s \in S$, $n \geq j + k$. If $n < j + k$, there is at most one $s \in S$ such that $B(x, 1/2^{j+k}) \cap V_{s,n} \neq \emptyset$, because $\mathcal{V}_n$ is discrete, as shown in the previous paragraph. Hence, $B(x, 1/2^{j+k})$ is a neighborhood of x meeting at most $j + k - 1$ members of the family $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$.